Quantitative Density under Higher Rank Abelian Algebraic Toral Actions

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Abstract

We generalize Bourgain-Lindenstrauss-Michel-Venkatesh's recent one-dimensional quantitative density result to abelian algebraic actions on higher dimensional tori. Up to finite index, the group actions that we study are conjugate to the action of U_K, the group of units of some non-CM number field K, on a compact quotient of $K \otimes_{\mathbb{Q}} \mathbb{R}$. In such a setting, we investigate how fast the orbit of a generic point can become dense in the torus. This effectivizes a special case of a theorem of Berend; and is deduced from a parallel measure-theoretical statement which effectivizes a special case of a result by Katok-Spatzier. In addition, we specify two numerical invariants of the group action that determine the quantitative behavior, which have number-theoretical significance.

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1 Introduction

1.1 Background

The rigidity of higher rank abelian algebraic actions has since long been studied in various forms. The first result of this type was achieved by Furstenberg's disjointness theory :

Theorem 1.1. (Furstenberg [8], '67) *Any minimal closed subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ simultaneously invariant under $\times 2$ and $\times 3$ is either $\mathbb{T}$ itself or a finite set of rational points.*

The measure-theorectical analogue of this theorem is the famous Furstenberg's conjecture, which asks whether any ergodic $\times 2, \times 3$-invariant measure on $\mathbb{T}$ is either uniform or finitely supported on rational points. The conjecture remains open.

There are several ways to extend Furstenberg's theorem. One of them is achieved by Berend, who proved an analogue on higher dimensional tori.

Theorem 1.2. (Berend [2], '83) *Let $\Sigma < M_d(\mathbb{Z}) = \mathrm{End}(\mathbb{T}^d)$ be an abelian subsemigroup such that: (i) for any common eigenspace V of Σ, there is an element g whose eigenvalue corresponding to V has absolute value strictly greater than 1; (ii) $\mathrm{rank}(\Sigma) \geq 2$; (iii) Σ contains a totally irreducible element. Then any minimal Σ-invariant closed set on $\mathbb{T}^d$ is either the full torus or finite.*

Here the action of a single toral endomorphism $A \in M_d(\mathbb{Z})$ is *irreducible* if there is no non-trivial A-invariant subtorus on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$; it is *totally irreducible* if A^n is irreducible for every positive integer n.

Another way of extension was to go to the measure-theoretical category. Under the assumption of positive entropy, Furstenberg's conjecture has been proved by Rudolph and Johnson following work of Lyons[17].

Theorem 1.3. (Rudolph[19],'90-Johnson[12],'92) *Suppose a Borel probability measure μ on $\mathbb{T}$ is invariant and ergodic under both $\times 2$ and $\times 3$. If the measure-theoretical entropy $h_\mu(\times 2)$ of the $\times 2$ action with respect to μ is positive, then μ is the Lebesgue measure.*

It is possible to pursue these two directions of extension at the same time. Namely, under certain conditions, for an abelian action on $\mathbb{T}^d$ with some kind of hyperbolicity, higher rank and irreducibility, any ergodic invariant measure with positive entropy is expected to be uniform. This was first proved by Katok and Spatzier[14] for abelian subgroups of $SL_d(\mathbb{Z})$ under a special assumption called total non-symplecticity (TNS); for a detailed treatment of their result, see Kalinin-Katok[13, §3]. Later on, such a measure rigidity statement was proved by Einsiedler-Lindenstrauss[6] for a more general family of groups of toral automorphisms.

Further, there is a third way to extend these results: all the theorems mentioned above have quantitative versions. Recently [3], Bourgain, Lindenstrauss, Michel and Venkatesh effectivized Furstenberg's theorem.

Theorem 1.4. (Bourgain-Lindenstrauss-Michel-Venkastesh [3],'08) *For any pair of multiplicatively independent positive integers a, b:*

(i). If $x \in \mathbb{T}$ is diophantine generic:

$$|x - \frac{p}{q}| \geq q^{-k},\ \forall p, q \in \mathbb{Z},\ q \geq 2,$$

then $\{a^m b^n x | 0 < m, n \leq N\}$ *is* $(\log\log Q)^{-c}$*-dense in* $\mathbb{T}$ *for all* $Q \geq Q_0$*, where* $c = c(a,b)$ *and* $Q_0 = N_0(k,a,b)$ *are constants.*

(ii). If $x = \frac{p}{Q}$ *where* Q *is coprime with* ab*, then* $\{a^m b^n x | 0 < m, n < 3\log Q\}$ *is* $C(\log\log\log Q)^{-c}$*-dense in* $\mathbb{T}$ *where* $C = C(a,b)$ *and* c *is the same as in (i).*

Roughly speaking, this says the orbit of a point x is of certain quantitative density unless x is very close to a rational number with small denominator.

1.2 Statement of main results

In this paper we generalize Bourgain-Lindenstrauss-Michel-Venkatesh's theorems to the higher dimensional case. We investigate a special case of the situation studied by Berend [2], namely the action on $\mathbb{T}^d$ by a group of toral automorphisms $G < SL_d(\mathbb{Z})$ that satisfies:

Condition 1.5. *G is an abelian subgroup of $SL_d(\mathbb{Z})$ such that:*

(1). $\operatorname{rank}(G) \geq 2$*, where* rank *refers to the torsion free rank of finitely generated abelian groups.;*

(2). G contains an totally irreducible toral automorphism;

(3). G is maximal in rank: there is no intermediate abelian subgroup G_1 in $SL_d(\mathbb{Z})$ containing G such that $\operatorname{rank}(G) < \operatorname{rank}(G_1)$.

We now equips G with a norm.

Definition 1.6. *(i). For a matrix $A \in M_d(\mathbb{R})$, we define the* **logarithmic Mahler measure** *of A to be*

$$m(A) := \frac{1}{2\pi}\int_0^{2\pi} \log|\det(A - e^{i\theta}\mathrm{id})|\mathrm{d}\theta.$$

An alternative definition is

$$m(A) = \sum_{i=1}^{d} \log_+ |\zeta_A^i|, \tag{1.1}$$

where $\zeta_A^1, \cdots, \zeta_A^d$ are the eigenvalues of A and $\log_+ x = \max(0, \log x)$.

(ii). For a subgroup $G < SL_d(\mathbb{Z})$ and $L > 0$, let $B_G^{\mathrm{Mah}}(L)$ be the ball of radius L with respect to logarithmic Mahler measure:

$$B_G^{\mathrm{Mah}}(L) := \{g \in G | m(g) \leq L\}.$$

For more information on Mahler measures, we refer to [7].

Here is our main result in the topological category:

Theorem 1.7. *If an abelian subgroup $G < SL_d(\mathbb{Z})$ satisfies Condition 1.5 then there are effective constants c_1, c_2, c_3, and c_4 depending only on G such that if a finite set $E \subset \mathbb{T}^d$ is ϵ-separated and $|E| \geq \epsilon^{-\alpha d}$ for some $\alpha, \epsilon > 0$ with $\epsilon \leq c_1$, $\alpha \geq c_2 \frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}$ then $B_G^{\mathrm{Mah}}(c_3 \log\frac{1}{\epsilon}).E := \{g.x | g \in B_G^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon}), x \in E\}$ is $(\log\frac{1}{\epsilon})^{-c_4\alpha}$-dense.*

The proof of Theorem 1.7 is based on the analogous effective measure-theoretical Theorem 1.9 below which studies the behaviour of a measure μ under the G-action. We assume μ has positive entropy up to a certain scale.

Condition 1.8. *μ is a Borel measure on $\mathbb{T}^d$ such that:*

For some given pair $\alpha > 0$, $\epsilon > 0$, the entropy $H_\mu(\mathcal{P}) = \sum_{P\in\mathcal{P}} -\mu(P)\log\mu(P)$ is at least $\alpha d \log\frac{1}{\epsilon}$ for all measurable partitions $\mathcal{P}$ of $\mathbb{T}^d$ such that $\mathrm{diam}\mathcal{P} \leq \epsilon$, where $\mathrm{diam}\mathcal{P}$ is the maximal diameter of atoms from $\mathcal{P}$.

Theorem 1.9. *Suppose an abelian subgroup $G < SL_d(\mathbb{Z})$ meets Condition 1.5. There are effective constants c_1, c_3, c_5, and c_6 depending only on G such that if $\epsilon \leq c_1$, $\delta \in [c_5(\log\log\frac{1}{\epsilon})^{-1}, \frac{\alpha}{2}]$ and a Borel measure μ on $\mathbb{T}^d$ satisfies the entropy condition 1.8 then there exists a measure μ' which has total mass $|\mu'| \geq \alpha - \delta$ and is dominated by some element μ'' from the convex hull of $B_G^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon}).\mu$ in the space of probability measures on $\mathbb{T}^d$ (here the group G acts on μ by pushforward: $g.\mu = g_*\mu$), so that $\forall f \in \mathcal{C}^\infty(\mathbb{T}^d)$,*

$$\left|\mu'(f) - |\mu'|\int_{\mathbb{T}^d} f(x)\mathrm{d}x\right| \leq c_6(\log\frac{1}{\epsilon})^{-\frac{1}{2}{c_5}^{-1}\delta}\|f\|_{\dot{H}^{\frac{d+1}{2}}}.$$

Theorem 1.9 actually effectivizes a special case of Katok-Spatzier's result [14].

Another interesting corollary to Theorem 1.7 will be Theorem 1.10, regarding how fast a generic single G-orbit can fill up $\mathbb{T}^d$, which is a quantitative form of the fact that any infinite G-orbit is dense (proved by Berend [2]) and the generalization of a similar theorem in [3].

Theorem 1.10. *Suppose G satisfies Condition 1.5 then there are effective constants c_7, c_8 depending only on G such that:*

For all $Q \in \mathbb{N}$, $Q \geq c_7$, if a point $x \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ satisfies one of the following conditions:

(i) x is diophantine generic: $\exists k > 1$ such that $|x - \frac{v}{q}| \geq q^{-k}$ for any coprime pair (v, q) where $q \in \mathbb{N}$ and $v \in \mathbb{Z}^d$; OR

(ii) $x = \frac{v}{Q}$ where $v \in \mathbb{Z}^d$ is coprime with Q, in which case we denote $k = 1$,

then the set $B_G^{\mathrm{Mah}}\big((k+2)\log Q\big).x$ is $(\log\log\log Q)^{-c_8}$-dense.

Remark 1.11. *In fact, in all the theorems stated above, we are able to know on which features of G the constants really depend and how the dependence looks like. Actually, all constants are determined by the dimension d and two algebraic invariants $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$ of G. For the explicit expressions, see Propositions 6.10, 7.1, 7.2, 7.6.*

The exact definitions of $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$ are going to appear later in §2. However, essentially $\mathcal{M}_\psi$ is a measurement of how "twisted" the eigenbasis of G is (notice G is commutative so all its elements share a common eigenbasis); and $\mathcal{F}_{\phi(G)}$ is the bound on a set of generators of G (up to finite-index) in terms of logarithmic Mahler measure.

We make an effort to track the dependence on $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$ in this paper, the reason is that this dependence can be interesting from a number-theoretical point of view (cf. Appendix §8 for example).

Remark 1.12. *It should be remarked that if $\|\cdot\|_{\mathrm{WL}}$ is the word length metric with respect to some fixed generating set $S \subset G$, then $m(g) \lesssim_{G,S} \|g\|_{\mathrm{WL}} + 1, \forall g \in G$. Therefore in all the theorems stated above, the ball $B_G^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon})$ with respect to Mahler measure can be replaced by some ball $B_G^{\mathrm{WL}}({c_3}'\log\frac{1}{\epsilon})$ defined in terms of word length metric, where ${c_3}' = {c_3}'(G, S)$ is effective.*

A major restricion we adopted in addition to Berend's conditions is that G is supposed to be maximal in rank, which guarantees that one can expand an arbitrary eigenspace while contracting everything else. The study of the case without this assumption is also possible, though more careful arguments are required and only weaker estimates will be obtained, and is hopefully going to appear in another paper [16].

1.3 Organization of paper and notations

While our proof follows the structure of that of Bourgain et. al. which uses Fourier analysis, we also borrow several arguments from [2]. A little bit of number theory is needed as our group G turns out to be closely linked to the full unit group of some number field. There are some new difficulties to be treated in the higher dimensional case. The main point is that instead of the case on a one dimensional circle, we are going to decompose a multidimensional torus into eigenspaces and work on one of them. We will show some kind of "dense" distribution along a line in that eigenspace and then make use of the irreduciblity of the action (so any eigenspace does not form a close subtorus). One of the difficulties here is that when the eigenspace is complex (i.e. 2-dimensional), a line in it may not be equidistributed in $\mathbb{T}^d$, this is going to be dealt with in §5.3

The organization of paper is as follows:

§2 discusses the number-theoretical implication of Condition 1.5 and proves the G-action comes from the group of units of a certain number field and its eigenspaces correspond to the real and complex embeddings of that field. Further, in §3 we discuss a few algebraic properties of this action, in particular the irrationality of the eigenspaces is specified quantitatively.

In §4 we give an effective description of the fact that if a measure on $\mathbb{T}^d$ has positive entropy then its projection to at least one of the eigenspaces V_i has positive entropy. Since the restriction of the group action on V_i is just a multiplication, we are able to discuss its behaviors in §5. Roughly speaking, we are going to show that using the group action, it is possible to stretch a short vector, to generate an approximation of a line segment (or an arithmetic progression) from a given vector, as well as to relocate an arbitrary line to a general position if the eigenspace is a complex one.

Proof of Theorem 1.9 is given in §6. The main tool used there is Fourier analysis. And the underlying geometric idea is that the positive entropy in V_i guarantees that the difference vector between a random pair of nearby points taken with respect to μ is not too short, and by applying the results obtained in §5 to this vector we can create a sufficiently long line segment placed in a general position, which is equidistributed on the torus.

In §7 Theorem 1.7 is deduced from 1.9 by taking a test function, we then prove its corollaries. The appendix §8 gives an application of results in §7 to a number theoretical problem, following a strategy observed by Cerri [5].

Notation 1.13. *Throughout this paper,*

- $\log$ *stands for the logarithm of base* 2 *and the natural logarithm is denoted by* $\ln$.
- *For a linear map f, $\|f\| := \sup_{|x|=1} |f(x)|$ is its norm as an operator.*
- *If A and B are subsets of some additive group, let $A - B$ (resp. $A + B$) denote the set $\{a - b \text{ (resp. } a + b) \,|a \in A, b \in B\}$.*
- *The symbol* rank *refers to the torsion-free rank of a finitely generated abelian group.*
- *We denote constants by $c_1, c_2, \ldots$ and less important ones by $\kappa_1, \kappa_2, \ldots$. The constants only locally used by a proof are written as $\iota_1, \iota_2, \ldots$. All constants in this paper are going to be positive and effective: i.e. an explicit value for it can be deduced from the information already known if one really want to. When a constant first appears, we may write it as a function to signify all the variables that it depends on, for example if we write $c_3(N, d)$, it means that the constant c_3 depends only on N and d, and so forth.*

- *We write $A \lesssim B$ (or $B \gtrsim A$) for the estimate $\exists c > 0, A \leq cB$. Moreover, we always include all factors that the implied constant c depends on within the $\lesssim$ symbol as subscripts. For example, if c depends on and only on d then we will always write $A \lesssim_d B$; the inequality $A \lesssim B$ without any subscript means the implied constant is absolute.*

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2 Abelian algebraic actions on the torus

In this section, we are going to impose an alternative condition on G to replace Condition 1.5.

2.1 Preliminaries on number fields

Consider now a degree d number field K, $d \geq 3$. $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$. Any element of K acts on this space naturally by multiplication: $\times_s.(t \otimes x) = st \otimes x, \forall s, t \in K, x \in \mathbb{R}$. Recall that if K has r_1 real embeddings $\sigma_1, \cdots, \sigma_{r_1}$ and r_2 pairs of conjugate imaginary embeddings $\sigma_{r_1+j}, \sigma_{r_1+r_2+j}, j = 1, \cdots, r_2$, then there is an isomorphism $\sigma : K \otimes_{\mathbb{Q}} \mathbb{R} \mapsto \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^d$ where

$$\begin{aligned}\sigma(t \otimes x) =& x \cdot \big(\sigma_1(t), \cdots, \sigma_{r_1}(t), \mathrm{Re}\sigma_{r_1+1}(t), \cdots, \mathrm{Re}\sigma_{r_1+r_2}(t), \\ & \mathrm{Im}\sigma_{r_1+1}(t), \cdots, \mathrm{Im}\sigma_{r_1+r_2}(t)\big)\end{aligned} \tag{2.1}$$

for all $x \in \mathbb{R}$, $t \in K$. With this identification, the action of K on $\mathbb{R}^d \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ is easy to describe: $\times_s$ acts on $K \otimes_{\mathbb{Q}} \mathbb{R}$ via the linear mapping $(\times_{\sigma_1(s)}, \cdots, \times_{\sigma_{r_1}(s)}, \times_{\sigma_{r_1+1}(s)}, \cdots, \times_{\sigma_{r_1+r_2}(s)})$, where the first r_1 multiplications are on the r_1 real subspaces and the last r_2 ones are on the complex ones. Here we view the j-th copy of $\mathbb{C}$ as a two-dimensional real subspace. In addition, let it be spanned by the $(r_1 + j)$-th (real part) and the $(r_1 + r_2 + j)$-th (imaginary part) coordinates: a number z in this copy of $\mathbb{C}$ is identified with $(\mathrm{Re}z)e_{r_1+j} + (\mathrm{Im}z)e_{r_1+r_2+j}$, then the multiplicative action of s on this embedded copy of $\mathbb{C}$ is given by the matrix $\begin{pmatrix} \mathrm{Re}\sigma_{r_1+j}(s) & -\mathrm{Im}\sigma_{r_1+j}(s) \\ \mathrm{Im}\sigma_{r_1+j}(s) & \mathrm{Re}\sigma_{r_1+j}(s) \end{pmatrix}$.

Let $\mathcal{O}_K$ be the ring of integers and $U_K = \mathcal{O}_K^*$ be the group of units of K. Dirichlet's Unit Theorem claims that, modulo the torsion part $\mathcal{T}_k$ of U_k, which is a finite set of roots of unity, the group morphism

$$\mathcal{L} : t \to \big(\log|\sigma_1(t)|, \log|\sigma_2(t)|, \cdots, \log|\sigma_{r_1+r_2}(t)|\big) \tag{2.2}$$

embeds U_K as a lattice in the $(r_1 + r_2 - 1)$-dimensional subspace

$$W = \{(w_1, w_2, \cdots, w_{r_1+r_2}) | \sum_{j=1}^{r_1} w_j + 2\sum_{j=1}^{r_2} w_{r_1+j} = 0\} \subset \mathbb{R}^{r_1+r_2}. \tag{2.3}$$

Let $r = r_1 + r_2 - 1 = \mathrm{rank}(U_K)$ and $d_i = \begin{cases} 1, & 1 \leq i \leq r_1; \\ 2, & r_1 + 1 \leq i \leq r_1 + r_2. \end{cases}$ then $d = \sum_1^{r_1+r_2} d_i$.

Remark 2.1. *For a given d, there are only finitely many possible values for r as $\frac{d}{2} - 1 = \frac{r_1}{2} + r_2 - 1 \leq r = r_1 + r_2 - 1 \leq r_1 + 2r_2 - 1 = d - 1$.*

For an element in K, its logarithmic Mahler measure measures the size of its image under $\mathcal{L}$.

Definition 2.2. *For a non-zero element t from a number field K of degree d, the* **logarithmic Mahler measure** *of t is*

$$h^{\mathrm{Mah}}(t) = \sum_{\nu} d_\nu \log_+ |t|_\nu,$$

where $\log_+ x = \max(\log x, 0)$, ν runs over all finite or infinite places of K and $d_\nu = [K_\nu : \mathbb{Q}_{\tilde{\nu}}], \nu|\tilde{\nu}$ is the local degree of ν.

$h(t) = \dfrac{h^{\mathrm{Mah}}(t)}{d}$ is the **absolute logarithm height** *of t, it is determined by the algebraic number t and does not depend on the field K in which t is regarded as an element.*

In particular, if $t \in U_K$ then all its non-Archimedean absolute values are equal to 1, so the logarithmic Mahler measure involves only Archimedean embeddings:

$$h^{\mathrm{Mah}}(t) = \sum_{i=1}^{r_1+r_2} d_i \log_+ |\sigma_i(t)| = \frac{1}{2}\sum_{i=1}^{r_1+r_2} d_i \big|\log|\sigma_i(t)|\big| = \frac{1}{2}\sum_{i=1}^{d} |\log \sigma_i(t)| \tag{2.4}$$

because $\sum_{i=1}^{r_1+r_2} d_i \log|\sigma_i(t)| = \log|\prod_{i=1}^{d}\sigma_i(t)| = 0$.

It is easy to see

$$h^{\mathrm{Mah}}(tt') \leq h^{\mathrm{Mah}}(t) + h^{\mathrm{Mah}}(t'); \; h^{\mathrm{Mah}}(t^n) = |n| h^{\mathrm{Mah}}(t), \forall n \in \mathbb{Z}. \tag{2.5}$$

Definition 2.3. *Let U be a finite index subgroup in U_K. Define the* **size of (virtually) fundamental units** *to be*

$$\mathcal{F}_U := \inf \Big\{ \mathcal{F} | \{h^{\mathrm{Mah}}(t) \leq \mathcal{F}, t \in U\} \text{ generates a finite-index subgroup of } U \Big\}.$$

Notice as U is a discrete set, $\mathcal{F}_U$ can be achieved, i.e. there are r elements $t_1, \cdots, t_r$ that virtually generate the torsion-free part of U, such that $\max_k h^{\mathrm{Mah}}(t_k) = \mathcal{F}_U$.

Remark 2.4. *It is known that for all algebraic numbers t of degree d, $h^{\mathrm{Mah}}(t)$ has an effective positive lower bound $\kappa_1(d)$ depending only on d as long as t is not a root of unity (see Voutier [21]). By definition this is also a lower bound for $\mathcal{F}_U$.*

We are interested in lattices embedded by σ into $K \otimes_{\mathbb{Q}} \mathbb{R}$, the covolumes and shapes of such lattices will be important for us.

Definition 2.5. *(1). The* **scale** *of a lattice Λ in a d-dimensional vector space V is $\mathcal{S}_\Lambda = \sqrt[d]{\mathrm{covol}(\Lambda)}$.*

(2). The **uniformity** *of a linear isomorphism $\Psi : \mathbb{R}^d \mapsto \mathbb{R}^d$ is*

$$\begin{aligned}\mathcal{M}_\Psi :=& \max(\|\Psi\| \mathcal{S}^{-1}_{\Psi(\mathbb{Z}^d)}, \|\Psi^{-1}\| \mathcal{S}_{\Psi(\mathbb{Z}^d)}) \\ =& \max(\|\Psi\| \cdot |\det \Psi|^{-\frac{1}{d}}, \|\Psi^{-1}\| \cdot |\det \Psi|^{\frac{1}{d}}).\end{aligned}$$

Definition 2.6. *A number field K is a* **CM-number field** *if it has a proper subfield F such that $\mathrm{rank}(U_K) = \mathrm{rank}(U_F)$.*

Actually, it follows from Dirichlet's Unit Theorem that a number field is CM if and only if K is a totally complex quadratic extension of a totally real field F (see Parry [18]).

We show a property of non-CM fields that is going to be relevant later.

Lemma 2.7. *If K is not CM and has a complex embedding $\sigma_{r_1+j}, 1 \le j \le r_2$, let F be the proper subfield $\sigma_{r_1+j}^{-1}(\mathbb{R})$. If $F \neq \mathbb{Q}$ then $\exists k, l \in \{1, \cdots, r_1+r_2\} \backslash \{r_1+j\}$ such that $k \neq l$ but $\forall t \in F$, $|\sigma_k(t)| = |\sigma_l(t)|$.*

Proof. Let $d' = [K : F] \ge 2$, then each embedding of F extends to exactly d' different embeddings of K. In other words, the set of embeddings $\{\sigma_1, \cdots, \sigma_d\}$ of K can be divided into d'-tuples, each corresponds to one embedding of F; the number of these tuples is $[F : \mathbb{Q}] = \frac{d}{d'}$.

If $d' = 2$, each tuple is a pair. As σ_{r_1+j} and its complex conugate $\sigma_{r_1+r_2+j}$ has the same restriction on the maximal real subfield $F = \sigma_{r_1+j}^{-1}(\mathbb{R})$, they form one of the pairs. So any other pair does not contain σ_{r_1+j} or $\sigma_{r_1+r_2+j}$. Suppose every other pair consists of two conjugate complex embeddings σ and $\bar{\sigma}$ then as they have the same restriction on F, F lies in their common real part. So all embeddings of F is real and extends to two conjugate complex embeddings of K, which contradicts the assumption that K is not CM. Hence there is a pair made of two embeddings which are not complex conjugates to each other.

If $d' \ge 3$, then one of the tuple contains σ_{r_1+j} and $\sigma_{r_1+r_2+j}$. As $F \neq \mathbb{Q}$, there is at least one other d'-tuple. As $d' \ge 3$, there are two embeddings in that tuple that are not complex conjugate to each other.

So in any case there are two different embeddings σ and σ' of K, which are not σ_{r_1+j} or $\sigma_{r_1+r_2+j}$ and are not complex conjugate to each other, such that $\sigma|_F = \sigma'|_F$. If one or both of them is not in $\sigma_1, \cdots, \sigma_{r_1+r_2}$ then we replace it by its complex conjugate, which does not change the absolute value. This completes the proof. $\square$

Condition 2.8. *G is an abelian subgroup of $SL_d(\mathbb{Z})$ such that there are:*

(1). a non-CM number field K of degree d whose group of units U_K has rank at least 2;

(2). an isomorphim ϕ from G to a finite-index subgroup of U_K;

(3). a cocompact lattice Γ in $K \otimes_{\mathbb{Q}} \mathbb{R}$ which sits in $K < K \otimes_{\mathbb{Q}} \mathbb{R}$ and is invariant under the natural action of $\phi(G)$,

such that by identifying G with $\phi(G)$, the action of G on $\mathbb{T}^d$ is conjugate to the natural G-action on $X = (K \otimes_{\mathbb{Q}} \mathbb{R})/\Gamma$, i.e., there exists a linear isomorphism $\psi : \mathbb{R}^d \mapsto K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ such that $\psi(\mathbb{Z}^d) = \Gamma$ and $\psi^{-1} \circ \times_{\phi(g)} \circ \psi = g, \forall g \in G$.

Remark 2.9. $\mathrm{rank}(U_K) = r_1 + r_2 - 1 \ge 2$ *implies* $d = r_1 + 2r_2 \ge 3$.

A consequence to Condition 2.8 is that the logarithmic Mahler measures defined respectly by Definition 1.6 on G and by Definition 2.2 on $\phi(G)$ are identified via ϕ.

Lemma 2.10. *If G satisfies Condition 2.8, then $\forall g \in G$, the logarithmic Mahler measure $m(g) = h^{\mathrm{Mah}}(\phi(g))$.*

Proof. The eigenvalues of g are exactly those of $\times_{\phi(g)}$, namely $\sigma_1(\phi(g)), \cdots, \sigma_d(\phi(g))$. By (1.1), $m(g) = \sum_i \log_+ \sigma_i(\phi(g)) = h^{\mathrm{Mah}}(\phi(g))$. $\square$

Remark 2.11. *If a generating set S of G is fixed, by Dirichlet's Unit Theorem it is not difficult to see that the word metric $\|g\|^{\mathrm{WL}}$ of an arbitrary element g with respect to S has some upper bound $\|g\|^{\mathrm{WL}} \lesssim_{G,S} h^{\mathrm{Mah}}(\phi(g)) + 1$, from which Remark 1.12 follows.*

It turns out Condition 2.8 is a good substitute for Condition 1.5.

Theorem 2.12. *Condition 1.5 implies Condition 2.8.*

Proof. The theorem decomposes into Propositions 2.13, 2.14 and 2.16, which are proved below. $\square$

In fact, it can be shown that the two conditions are equivalent.

2.2 Construction of the number field

In the following three subsections we are going to prove Theorem 2.12. The following statement is a special case of a known theorem (see for example [6, Prop. 2.1], , for a proof of the general statement see [20]).

Proposition 2.13. *Suppose an abelian subgroup G of $SL_d(\mathbb{Z})$ has an element whose action is irreducible, then there exist*

(i). a number field K of degree d;

(ii). an isomorphim ϕ from G to a subgroup of the group of units U_K;

(iii). a rank d lattice Γ in $K < K \otimes_{\mathbb{Q}} \mathbb{R}$ which is invariant under the natural action of $\phi(G)$,

such that there exists a linear isomorphism $\psi : \mathbb{R}^d \mapsto K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ such that $\psi(\mathbb{Z}^d) = \Gamma$ and $\psi^{-1} \circ \times_{\phi(g)} \circ \psi = g, \forall g \in G$;

Proof of Proposition. Assume $g \in G$ acts irreducibly, then the characteristic polynomial of g is irreducible over $\mathbb{Q}$; otherwise the rational canonical form of g over $\mathbb{Q}$ has more than one block, giving a non-trivial g-invariant rational subspace of $\mathbb{R}^d$, which projects to a non-trivial g-invariant subtorus in $\mathbb{T}^d$. Therefore g has d distinct eigenvalues $\zeta_g^1, \cdots, \zeta_g^d$ which are algebraic conjugates to each other, where $\zeta_g^1, \cdots, \zeta_g^{r_1}$ are real and the rest are r_2 imaginary pairs, $r_1 + 2r_2 = d$. $\zeta_g^{r_1+j} = \overline{\zeta_g^{r_1+r_2+j}}, \forall j = 1, \cdots, r_2$.

Construct number field $K_i = \mathbb{Q}(\zeta_g^i)$, so $\deg K_i = d$. Denote $K = K_1$, then K has d embeddings: $\sigma_i(\zeta_g^1) = \zeta_g^i, \sigma_i(K) = K_i, \forall i = 1, \cdots, d$. The first r_1 embeddings are real; σ_{r_1+j} and $\sigma_{r_1+r_2+j}$ are complex conjugates for $1 \leq j \leq r_2$.

An eigenvector v_1 of ζ_g^i lies in $\mathrm{Ker}(g - \zeta_g^1 \mathrm{id})$. As all entries of $g - \zeta_g^1\mathrm{id}$ belongs to the field K, we can fix an eigenvector $v_1 \in K^d$.

Let $v_i = \sigma_i(v_1) \in {K_i}^d$, then as $g \in SL_d(\mathbb{Z})$ is fixed by σ_i, $gv_i = \sigma_i(g)\sigma_i(v_1) = \sigma_i(\zeta_g^1 v_1) = \sigma_i(\zeta_g^1)\sigma_i(v_1) = \zeta_g^i v_i$. So v_i is an eigenvector corresponding to the eigenvalue ζ_g^i for all $1 \leq i \leq d$ and it follows that $\mathbb{C}^n = \oplus_{i=1}^d \mathbb{C} v_i$. Notice $v_i \in \sigma_i(K^d) = K_i^d \subset \mathbb{C}^d$, in particular it is a real vector for $1 \leq i \leq r_1$.

As g is irreducible all eigenspaces are one dimensional over $\mathbb{C}$, thus by commutativity the basis $\{v_i\}_{i=1}^d$ diagonalizes not only g but any element $h \in G$ as well: $hv_i = \zeta_h^i v_i, \forall h \in G, \forall i$.

We claim $\zeta_h^i \in K_i, \forall h \in G$. This is because an arbitrary element t in the field $\mathbb{Q}(\zeta_g^i, \zeta_h^i)$ can be written as $p(\zeta_g^i, \zeta_h^i)$ where p is a rational polynomial. As $gh = hg$, $p(g,h)v_i = p(\zeta_g^i, \zeta_h^i)v_i = tv_i$, so t is an eigenvalue of the rational $d \times d$ matrix $p(g,h)$, thus an algebraic number of degree at most d. By choosing t to be a generating element of $\mathbb{Q}(\zeta_g^i, \zeta_h^i) = K_i(\zeta_h^i)$, $\deg K_i(\zeta_h^i) \leq d = \deg K_i$. Thus $\zeta_h^i \in K_i$.

Furthermore, $\sigma_i(\zeta_h^1) = \zeta_h^i$ for any $h \in G$. Actually, as $\zeta_h^1 \in K$, $\zeta_h^1 = f(\zeta_g^1)$ for some polynomial $f \in \mathbb{Q}[x]$. $f(g)v_1 = f(\zeta_g^1) = \zeta_h^1 v_1 = hv_1$. If $f(g) \neq h$ then $0 < \mathrm{rank}(f(g) - h) < d$, $\mathrm{Ker}(f(g) - h)$ is a non-trivial rational subspace of $\mathbb{R}^d$. Since g commutes with $f(g) - h$, this rational subspace is g-invariant, which is prohibited to happen as g acts irreducibly. So $h = f(g)$, in particular $\sigma_i(\zeta_h^1) = \sigma_i(f(\zeta_g^1)) = f(\sigma_i(\zeta_g^1)) = f(\zeta_g^i) = \zeta_h^i$.

Let $\phi(h) = \zeta_h^1$, which is an algebraic integer as h is an integer matrix. Moreover $\zeta_h^1 \in U_K$ because h is invertible. Hence ϕ is a group morphism whose image lies in U_K. ϕ is injective because if $\phi(h) = 1$ for some $h \neq \mathrm{Id}$ then $\mathrm{Ker}(h - \mathrm{Id})$ is a non-trivial rational subspace invariant under g since g commutes with h, which again contradicts the irreducibility of g. Therefore $\phi : G \mapsto \phi(G) \subset U_K$ is injective.

For $1 \leq i \leq r_1$ let $w_i = v_i \in \mathbb{R}^d$; for $1 \leq j \leq r_2$, denote $w_{r_1+j} = 2\mathrm{Re}v_{r_1+j} = v_{r_1+j} + v_{r_1+r_2+j}$ and $w_{r_1+r_2+j} = -2\mathrm{Im}v_{r_1+j} = \mathrm{i}(v_{r_1+j} - v_{r_1+r_2+j})$. Then the real vectors $w_1, w_2, \cdots, w_d$ are linearly independent and thus form a basis of $\mathbb{R}^d \subset \mathbb{C}^d$.

Let ψ be the isomorphism from $\mathbb{R}^d \subset \mathbb{C}^d$ to $K \oplus_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d$ which maps w_i to e_i for all $1 \leq i \leq d$, where e_i is the i-th coordinate vector in $K \otimes_{\mathbb{Q}} \mathbb{R}$ which is represesented as in (2.1).

For any $h \in G$ acting on $\mathbb{R}^d$, it rescales w_i by $\zeta_h^i = \sigma_i(\phi(h))$ when $1 \leq i \leq r_1$ and acts on the two-dimensional subspace generated by $w_{r_1+j}, w_{r_1+r_2+j}$ as the matrix $\begin{pmatrix} \mathrm{Re}\sigma_{r_1+j}(\phi(h)) & -\mathrm{Im}\sigma_{r_1+j}(\phi(h)) \\ \mathrm{Im}\sigma_{r_1+j}(\phi(h)) & \mathrm{Re}\sigma_{r_1+j}(\phi(h)) \end{pmatrix}$. Compare this action with that of $\times_{\phi(h)}$ on $K \otimes_{\mathbb{Q}} \mathbb{R}$ which was described earlier, we see they are conjugate to each other via ψ.

Therefore the only fact remaining to show is that the lattice $\Gamma = \psi(\mathbb{Z}^d)$ is in $K \subset K \otimes_{\mathbb{Q}} \mathbb{R}$, which is equivalent to $\psi(\mathbb{Q}^d) = K$ as K is a d-dimensional $\mathbb{Q}$-vector space in $K \otimes_{\mathbb{Q}} \mathbb{R}$. It suffices to prove $\psi^{-1}(t) \in \mathbb{Q}^d$, $\forall t \in K$. Actually by description (2.1), t is identified with

$$\sum_{i=1}^{r_1} \sigma_i(t) e_i + \sum_{j=1}^{r_2} (\mathrm{Re}\sigma_{r_1+j}(t) \cdot e_{r_1+j} + \mathrm{Im}\sigma_{r_1+j}(t) \cdot e_{r_1+r_2+j}).$$

Using $\psi^{-1}(e_i) = w_i$, we obtain

$$\begin{aligned}
\psi^{-1}(t) =& \sum_{i=1}^{r_1} \sigma_i(t) w_i + \sum_{j=1}^{r_2} (\mathrm{Re}\sigma_{r_1+j}(t) \cdot w_{r_1+j} + \mathrm{Im}\sigma_{r_1+j}(t) \cdot w_{r_1+r_2+j}) \\
=& \sum_{i=1}^{r_1} \sigma_i(t) v_i + \sum_{j=1}^{r_2} \big(\mathrm{Re}\sigma_{r_1+j}(t)(v_{r_1+j} + v_{r_1+r_2+j}) \\
& \qquad\qquad + \mathrm{Im}\sigma_{r_1+j}(t) \cdot \mathrm{i}(v_{r_1+j} - v_{r_1+r_2+j})\big) \\
=& \sum_{i=1}^{r_1} \sigma_i(t) v_i + \sum_{j=1}^{r_2} (\sigma_{r_1+j}(t) v_{r_1+j} + \overline{\sigma_{r_1+j}}(t) v_{r_1+r_2+j}).
\end{aligned}$$

Since $\overline{\sigma_{r_1+j}}$ is just $\sigma_{r_1+r_2+j}$, we see

$$\psi^{-1}(t) = \sum_{i=1}^{d} \sigma_i(t) v_i = \sum_{i=1}^{d} \sigma_i(t) \sigma_i(v_1) = \sum_{i=1}^{d} \sigma_i(t v_1).$$

As $t \in K$ and $v_1 \in K^d$, each coordinate in the vector $\sum_{i=1}^{d} \sigma_i(t v_1)$ is the trace of the corresponding coordinate in $t v_1 \in K^d$, which is rational. This completes the proof. $\square$

2.3 Maximal rank and the full group of units

Proposition 2.14. *In Proposition 2.13, if there doesn't exist any abelian subgroup G_1 of $SL_d(\mathbb{Z})$ containing G such that* $\mathrm{rank}(G) < \mathrm{rank}(G_1)$*, then $\phi(G)$ is a subgroup of finite index inside U_K.*

Proof. Recall that in the proof of Proposition 2.13, all elements in G are diagonalized as a complex matrix with respect to basis $v_1, \cdots, v_d$. For any $\theta \in K$, construct a matrix $\gamma(\theta)$ which can be diagonalized in the same basis, such that $\gamma(\theta)v_i = \sigma_i(\theta)v_i, \forall i = 1, \cdots, d$, where $\sigma_i : K \mapsto K_i$ is the i-th embedding of K. It is clear that γ is an injective ring homomorphism between K and $M_d(\mathbb{C})$. Remark as well that $\forall h \in G$, $\gamma(\phi(h)) = h$.

As K is generated by $\phi(g) = \zeta_g^1$ where $g \in G$ is an irreducible element, $\forall \theta \in K$, θ can be expressed as a rational polynomial in $\phi(g)$. Hence $\gamma(\theta)$ can be written as the same rational polynomial in terms of $g = \gamma(\phi(g))$. So $\gamma(\theta) \in M_d(\mathbb{Q})$ because $g \in G < SL_d(\mathbb{Z})$. Let $\theta_1, \theta_2, \cdots, \theta_d$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$, the ring of integers, then there must be an integer D such that $\gamma(\theta_k) \in M_d(\frac{1}{D}\mathbb{Z}), \forall k = 1, \cdots, d$. Therefore for all $\theta \in \mathcal{O}_K$, $\gamma(\theta)$, which is a $\mathbb{Z}$-span of $\gamma(\theta_k)$'s, lies in $M_d(\frac{1}{D}\mathbb{Z})$ as well.

Moreover $\det(\gamma(\theta)) = N(\theta) = 1$ when $\theta \in U_K$, thus γ is an isomorphism between U_K and a subgroup in $SL_d(\frac{1}{D}\mathbb{Z})$.

For all $\theta \in U_K$, consider the sequence of lattices $\{\gamma(\theta)^n.\mathbb{Z}^d | n = 1, 2, \cdots\}$. For any n, $\gamma(\theta)^n.\mathbb{Z}^d$ has determinant 1 and is a sublattice of $(\frac{1}{D}\mathbb{Z})^d$ (because $\mathbb{Z}^d < (\frac{1}{D}\mathbb{Z})^d$ and $(\gamma(\theta))^n = \gamma(\theta^n) \in M_d(\frac{1}{D}\mathbb{Z})$), in other words it is a sublattice of index D^d in $(\frac{1}{D}\mathbb{Z})^d$. But there are only a finite number of sublattices of given index in a given lattice, hence there are $n_1 \neq n_2$ such that $\gamma(\theta)^{n_1}.\mathbb{Z}^d = \gamma(\theta)^{n_2}.\mathbb{Z}^d$. Which implies $\gamma(\theta)^n.\mathbb{Z}^d = \mathbb{Z}^d$ for $n = n_1 - n_2$, or equivalently, $\gamma(\theta)^n \in SL_d(\mathbb{Z})$.

So we have actually proved, for any element $\gamma(\theta)$ in the finitely generated abelian group $\gamma(U_K) \cong U_K$, there is an exponent n such that $(\gamma(\theta))^n \in SL_d(\mathbb{Z}) \cap \gamma(U_K)$. It follows that $\operatorname{rank}(\gamma(U_K) \cap SL_d(\mathbb{Z})) = \operatorname{rank}(\gamma(U_K)) = \operatorname{rank}(U_K)$.

By Proposition 2.13, $G = \gamma(\phi(G)) \subset \gamma(U_K) \cap SL_d(\mathbb{Z})$. Moreover, by assumption $\operatorname{rank}(G) = \operatorname{rank}\big(\gamma(U_K) \cap SL_d(\mathbb{Z})\big)$, therefore $\operatorname{rank}(G) = \operatorname{rank}(U_K)$. Since $G \cong \phi(G) < U_K$, this yields the proposition. $\square$

2.4 Total irreduciblity and non-CM number fields

Now we deal with the assumption that $G < SL_d(\mathbb{Z})$ contains at least one totally irreducible element.

Lemma 2.15. *If $g \in G$ is an irreducible toral automorphism, then $\phi(g)$ doesn't belong to any non-trivial proper subfield F of K.*

Proof. Any subfield F, viewed as a subset in $K \otimes_{\mathbb{Q}} \mathbb{R}$, is invariant under the multiplicative action of U_F. Assume $\phi(g) \in F$, then as the g-action on $\mathbb{R}^d$ is identified with that of $\phi(g)$ on $K \otimes_{\mathbb{Q}} \mathbb{R}$, g stablizes $\psi^{-1}(F)$. But in the proof of Proposition 2.13 we showed $\psi^{-1}(K) \subset \mathbb{Q}^d$, hence $\psi^{-1}(F)$ is a rational subspace in $\mathbb{Q}^d$ of dimension between 1 and $d-1$ because F is a non-trivial proper $\mathbb{Q}$-vector subspace in K. This contradicts the irreduciblity of g. $\square$

Proposition 2.16. *In Proposition 2.13, if G contains a totally irreducible toral automorphism then the number field K is not a CM-field.*

Proof. Assume there is a proper subfield F in K with $\operatorname{rank}(U_F) = \operatorname{rank}(U_K)$ for contradiction. Let g denote the totally irreducible element. Because the finite generated abelian groups U_K and U_F have the same rank, $\exists l \in \mathbb{N}$ such that $\phi(g^l) = \phi(g)^l \in U_K$ lies in U_F. The lemma above implies g^l is not irreducible, thus contradicts the total irreducibility. $\square$

This proves Theorem 2.12.

3 Action of the group of units

In the rest of this paper, we will always assume G, K, Γ, X, ϕ and ψ are as in Condition 2.8. In addition, two constants will play important roles in ours study: the uniformity $\mathcal{M}_\psi$ of ψ defined by Definition 2.5 and the size of fundamental units $\mathcal{F}_{\phi(G)}$ in $\phi(G) < U_K$ defined by Definition 2.3.

Notation 3.1. *$\forall 1 \leq i \leq d$, e_i denotes the i-th standard coordinate vector in $\mathbb{R}^d$, and by abuse of notation, those in the quotient $X = \mathbb{R}^d/\Gamma$ as well.*

For $1 \leq i \leq r_1$ let $V_i = \mathbb{R}e_i$ be the i-th real subspace in the product $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \cong \mathbb{R}^d$. And let $V_{r_1+j}, 1 \leq j \leq r_2$ be the j-th complex subspace (notice it is spanned by e_{r_1+j} and $e_{r_1+r_2+j}$). Then $\mathbb{R}^d = \oplus_{i=1}^{r_1+r_2} V_i$.

For $g \in G$ and $1 \leq i \leq d$, let $\zeta_g^i = \sigma_i(\phi(g))$, the i-th embedding of $\phi(g) \in U_K$. It is also the i-th eigenvalue of g, which is consistent with earlier notations.

As before, write $r = r_1 + r_2 - 1 = \mathrm{rank}(U_K) = \mathrm{rank}(G)$ and $d_i = \mathrm{dim}V_i, 1 \leq i \leq r$.

Recall $\forall g \in G$, if $1 \leq i \leq r_1$ then $\times_{\phi(g)}.e_i = \sigma_i(g)e_i$; if $1 \leq j \leq r_2$ then

$$\begin{cases} \times_{\phi(g)}.e_{r_1+j} & = (\mathrm{Re}\sigma_i(g))e_{r_1+j} + (\mathrm{Im}\sigma_i(g))e_{r_1+r_2+j}; \\ \times_{\phi(g)}.e_{r_1+r_2+j} & = (-\mathrm{Im}\sigma_i(g))e_{r_1+j} + (\mathrm{Re}\sigma_i(g))e_{r_1+r_2+j}. \end{cases}$$

3.1 Bounded totally irreducible units

We proved earlier K is not a CM-field from the total irreducibility. Conversely, a non-CM-field produces total irreducibility as well. Moreover, we can even construct a totally irreducible element with an explicit upper bound on its height.

Lemma 3.2. *Given $r \geq 2$, for any $N \in \mathbb{N}$, one needs at least $\kappa_2 N$ proper linear subspaces in $\mathbb{R}^r$ to cover the discrete set $I_N^r = \{-N, -N+1, \cdots, N-1, N\}^r \subset \mathbb{Z}^r$ where $\kappa_2 = \kappa_2(r)$ is effective.*

Proof. I_N^r is contained in the r-dimensional closed ball with radius $\sqrt{r}N$. The intersection of any proper linear subspace with this ball is a ball of radius $\sqrt{r}N$ in dimension $r-1$ or less. The number of lattice points in such a ball is $O_r(N^{r-1})$. Hence the number of proper subspaces neccessary to cover I_N^r is at least $\frac{(2N+1)^r}{O_r(N^{r-1})} \gtrsim_r N$. Thus there is a κ_2 for which the lemma holds. $\square$

Proposition 3.3. *If Condition 2.8 holds then there is a totally irreducible element $g \in G$ such that $h^{\mathrm{Mah}}(\phi(g)) \leq \kappa_3 \mathcal{F}_{\phi(G)}$ for some effective constant $\kappa_3(d)$.*

Proof. By definition of $\mathcal{F}_{\phi(G)}$, there are $g_1, \cdots, g_r \in G$ such that the $\phi(g_k)$'s generate a finite-index subgroup of $\phi(G)$ and $h^{\mathrm{Mah}}(\phi(g_k)) \leq \mathcal{F}_{\phi(G)}$. Let $I_N^G = \{\prod_{i=1}^r g_j^{N_j} | -N \leq N_j \leq N\}$ for some $N \geq 1$. Then $I_N^W := \{\mathcal{L}(\phi(g)) | g \in I_N^G\} = \{\sum_{i=1}^r N_j \mathcal{L}(\phi(g_j)) | -N \leq N_j \leq N\}$ is a discrete subset in W. It is obvious that the $\mathcal{L}(\phi(g_j))$'s span W. So I_N^W is linearly isomorphic to I_N^r and requires at least $\kappa_2 N$ proper subspaces of W to cover.

It follows from Galois theory that there is an explicit constant $\iota_1(d)$ such that the degree d number field K can have at most $\iota_1(d)$ proper subfields. For each subfield F, by the assumption that K is not a CM-field, U_F is an abelian subgroup in U_K of rank $r-1$ or less. Hence the $\mathbb{R}$-span of $\mathcal{L}(U_F)$ is a proper linear subspace in W. Take $N = \lfloor \frac{\iota_1}{\kappa_2} \rfloor + 1$, then $\kappa_2 N > \iota_1$ and there exists $g \in I_N^G$ such that $\mathcal{L}(\phi(g)) \notin \mathbb{R}\mathcal{L}(U_F)$ for all proper subfields F. So $\forall n \in \mathbb{N}$, $\forall F$,

$\mathcal{L}(\phi(g^n)) = n\mathcal{L}(\phi(g)) \notin \mathcal{L}(U_F)$ and thus $\phi(g^n) \notin F$, namely $\phi(g^n)$ is an irreducible element in K. Thus all eigenvalues of g^n, which are embeddings of $\phi(g^n)$, are of degree d. As a matrix in $SL_d(\mathbb{Z})$ is irreducible if and only if its eigenvalues are conjugate to each other and have degree d, g^n is irreducible for all $n \in \mathbb{N}$ and g is totally irreducible.

$h^{\mathrm{Mah}}(\phi(g)) \leq \sum_{j=1}^{r} N_j h^{\mathrm{Mah}}(\phi(g_j)) \leq rN\mathcal{F}_{\phi(G)}$. As $N = \lfloor \frac{\iota_1}{\kappa_2} \rfloor + 1$ explicitly depends on d and r, we may take $\kappa_3 = rN$, which depends only on d and r. But by Remark 2.1 κ_3 can be made independent of r by taking maximum over all possible values of r when d is fixed. □

3.2 Characters on X and irrationality of eigenspaces

$X = \mathbb{R}^d/\Gamma$ can identified with $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by ψ. Notice the determinant of ψ is exactly $\mathrm{covol}(\Gamma) = \mathrm{vol}(X) = \mathcal{S}_\Gamma^d$.

A character $\xi : X \mapsto \mathbb{R}/\mathbb{Z}$ is associated to the character $\mathbf{q} \in \mathbb{Z}^d = (\mathbb{T}^d)^*$ of $\mathbb{T}^d$ such that $\xi = \psi_*\mathbf{q}$, i.e. $\forall x \in X$, $\xi(x) = \langle \mathbf{q}, \psi^{-1}(x)\rangle$ where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. This association is bijiective.

Definition 3.4. *Given ψ with $\psi(\mathbb{Z}^d) = \Gamma$, for a character $\xi = \psi_*\mathbf{q} \in X^*$, denote $|\xi| = |\mathbf{q}|$ where $\mathbf{q} \in \mathbb{Z}^d$.*

For all $1 \leq i \leq r_1 + r_2$, we define a vector $\tilde{e}_i \in V_i \otimes_{\mathbb{R}} \mathbb{C} \subset \mathbb{C}^n$ by:

$$\begin{cases} \tilde{e}_i & = e_i, & 1 \leq i \leq r_1; \\ \tilde{e}_{r_1+j} & = \dfrac{\sqrt{2}}{2}(e_{r_1+j} - \mathrm{i}e_{r_1+j}), & 1 \leq j \leq r_2. \end{cases} \tag{3.1}$$

It is not hard to see $|\tilde{e}_i| = 1$ and $\times_{\phi(g)}.\tilde{e}_i = \zeta_g^i \tilde{e}_i, \forall g, i$.

Recall when $V \cong \mathbb{R}$ or $\mathbb{C}$ is regarded as a real vector space and $f : V \mapsto \mathbb{R}$ be a real linear form on it, if $V \cong \mathbb{R}$ then $f(v) = f_0 v$ for some $f_0 \in \mathbb{R}$ and $\|f\| = |f_0|$; and if $V \cong \mathbb{C}$ then $f(v) = f_1\mathrm{Re}v + f_2\mathrm{Im}v, f_1, f_2 \in \mathbb{R}$ and $\|f\| = \sqrt{f_1^2 + f_2^2}$.

The following result characterizes quantitatively the irrationality of the V_i's inside X.

Proposition 3.5. *If Condition 2.8 holds, then for all V_i and any non-trivial character $\xi \in X^*$,*

$$\left\|\xi\big|_{V_i}\right\| \geq d^{-1}2^{-\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}_\psi^{-(d-1)}|\xi|^{-(d-1)}\mathcal{S}_\Gamma^{-1},$$

where $\xi|_{V_i}$ denotes the restriction of ξ to $V_i \subset \mathbb{R}^d$ and $\kappa_3 = \kappa_3(d)$ is as in Proposition 3.3.

Proof. Suppose $\xi = \psi_*\mathbf{q}$ for some $\mathbf{q} \in \mathbb{Z}^d$. We view $\xi \in (\mathbb{R}^d)^* \subset (\mathbb{C}^d)^* \cong \mathbb{C}^d$ as a complex linear functional as well. Then $\|\xi\|$ coincide with the usual l^2-norm on $(\mathbb{C}^d)^*$.

By definition, if $1 \leq i \leq r_1$ then $\left\|\xi\big|_{V_i}\right\| = |\xi(e_i)| = |\xi(\tilde{e}_i)|$ and for $1 \leq j \leq r_2$ we have

$$\begin{aligned} \left\|\xi\big|_{V_i}\right\| =& \sqrt{\xi^2(e_{r_1+j}) + \xi^2(e_{r_1+r_2+j})} \\ =& \sqrt{\xi^2(\sqrt{2}\mathrm{Re}\tilde{e}_{r_1+j}) + \xi^2(\sqrt{2}\mathrm{Im}\tilde{e}_{r_1+j})} \\ =& \sqrt{2}|\xi(\tilde{e}_{r_1+j})|. \end{aligned}$$

So it suffices to show $|\xi(\tilde{e}_i)| \geq d^{-1}2^{-\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}_\psi^{-(d-1)}|\xi|^{-(d-1)}\mathcal{S}_\Gamma^{-1}, \forall i$.

Take the irreducible element $g \in G$ in Proposition 3.3. Denote $a = |\xi(\tilde{e}_i)|$, then $|\zeta_g^i|^n a = |\xi((\zeta_g^i)^n \tilde{e}_i)| = |\xi(\times_{\phi(g)}^n . \tilde{e}_i)| = |\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi\big)(\tilde{e}_i)|$ for all n. This means if we write $(\times_{\phi(g)}^{\mathrm{T}})^n \xi = \xi_n^{\perp} + \xi_n$ where $\xi_n^{\perp}$ is the projection to the complex hyperplane $\tilde{e}_i^{\perp}$ and ξ_n is orthogonal to it, then $\|\xi_n\| \leq \frac{|\zeta_g^i|^n a}{|\tilde{e}_i|} = |\zeta_g^i|^n a$.

Consider the determinant $\det\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi\big)_{n=0}^{d-1}$. It can be written as

$$\det\big(\xi_n^{\perp}\big)_{n=0}^{d-1} + \sum_{m=0}^{d-1} \det\big(\xi_0^{\perp}, \cdots, \xi_{m-1}^{\perp}, \xi_m, (\times_{\phi(g)}^{\mathrm{T}})^{m+1}\xi, \cdots, (\times_{\phi(g)}^{\mathrm{T}})^{d-1}\xi\big). \tag{3.2}$$

$\det(\xi_n^{\perp})_{n=0}^{d-1} = 0$ as the $\xi_n^{\perp}$'s are all in the same hyperplane $\tilde{e}_i^{\perp}$ and for every index m, the absolute value of the corresponding term in the sum is at most

$$\begin{aligned}
&\|\xi_0^{\perp}\| \cdots \|\xi_{m-1}^{\perp}\| \cdot \|\xi_m\| \cdot \|(\times_{\phi(g)}^{\mathrm{T}})^{m+1}\xi\| \cdots \|(\times_{\phi(g)}^{\mathrm{T}})^{d-1}\xi\| \\
\leq &\|\xi\| \cdots \|(\times_{\phi(g)}^{\mathrm{T}})^{m-1}\xi\| \cdot \|\xi_m\| \cdot \|(\times_{\phi(g)}^{\mathrm{T}})^{m+1}\xi\| \cdots \|(\times_{\phi(g)}^{\mathrm{T}})^{d-1}\xi\| \\
\leq &\| \times_{\phi(g)}^{\mathrm{T}} \|^{\sum_{0\leq j\leq d-1, j\neq m} j} \|\xi\|^{d-1} \cdot |\zeta_g^i|^m a. \\
\leq &\| \times_{\phi(g)}^{\mathrm{T}} \|^{\sum_{j=0}^{d-1} j} \|\xi\|^{d-1} a.
\end{aligned}$$

Here we used the fact that $\| \times_{\phi(g)}^{\mathrm{T}} \| = \max_{1\leq j\leq d} \|\sigma_j(g)\| = \max_{1\leq j\leq d} \|\zeta_g^j\|$, which follows directly from the construction of $\times_{\phi(g)}$. Moreover,

$$\|\zeta_g^i\| \leq \| \times_{\phi(g)}^{\mathrm{T}} \| \leq 2^{\max_{1\leq j\leq d} \log \|\sigma_j(\phi(g))\|} \leq 2^{h^{\mathrm{Mah}}(\phi(g))} \leq 2^{\kappa_3 \mathcal{F}_\phi(G)}.$$

From (3.2) we obtain an upper bound

$$|\det\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi\big)_{n=0}^{d-1}| \leq d 2^{\frac{d(d-1)}{2}\kappa_3 \mathcal{F}_\phi(G)} \|\xi\|^{d-1} a. \tag{3.3}$$

However $(\times_{\phi(g)}^{\mathrm{T}})^n \xi = \psi_*\big((g^{\mathrm{T}})^n \mathbf{q}\big)$, so

$$\det\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi)\big)_{n=0}^{d-1} = \det(\psi^{-1}) \det\big((g^{\mathrm{T}})^n \mathbf{q}\big)_{n=0}^{d-1} = \mathcal{S}_\Gamma^{-d} \det\big((g^{\mathrm{T}})^n \mathbf{q}\big)_{n=0}^{d-1}.$$

But $\det\big((g^{\mathrm{T}})^n \mathbf{q}\big)_{n=0}^{d-1} \neq 0$. This is because if the determinant vanishes then the $\mathbb{Q}$-span of the vectors $\{(g^{\mathrm{T}})^n \mathbf{q}\}_{0\leq n\leq d-1}$ is a non-trivial proper rational subspace $L \subset (\mathbb{R}^d)^*$. But the characteristic polynomial of g^T is rational and has degree d by irreducibility, all powers of g^{T} are in the $\mathbb{Q}$-span of $\{(g^{\mathrm{T}})^n\}_{0\leq n\leq d-1}$. Whence for all $n \in \mathbb{Z}$, $(g^{\mathrm{T}})^n \mathbf{q} \in L$ and as a result, L is invariant under g^{T}. So $L^{\perp}$, which is a non-trivial proper subspace, is stablized by g. This contradicts the irreducibility of g. Therefore, the fact that $(g^{\mathrm{T}})^n \mathbf{q}$ is an integer vector for all n gives a lower bound $\big|\det\big((g^{\mathrm{T}})^n \mathbf{q}\big)_{n=0..d-1}\big| \geq 1$. So $\big|\det\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi\big)_{n=0}^{d-1}\big| \geq \mathcal{S}_\Gamma^{-d}$.

On the other hand, $\|\xi\| = \|\psi_* q\| \leq \|\psi^{-1}\| \cdot \|q\| \leq \mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} |q| = \mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} |\xi|$.

Thus it follows from (3.3) that

$$\begin{aligned}
a \geq &d^{-1} 2^{-\frac{d(d-1)}{2}\kappa_3 \mathcal{F}_\phi(G)} \|\xi\|^{-(d-1)} \big|\det\big((\times_{\phi(g)}^{\mathrm{T}})^n \xi\big)_{n=0}^{d-1}\big| \\
\geq &d^{-1} 2^{-\frac{d(d-1)}{2}\kappa_3 \mathcal{F}_\phi(G)} (\mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} |\xi|)^{-(d-1)} \mathcal{S}_\Gamma^{-d} \\
= &d^{-1} 2^{-\frac{d(d-1)}{2}\kappa_3 \mathcal{F}_\phi(G)} \mathcal{M}_\psi^{-(d-1)} |\xi|^{-(d-1)} \mathcal{S}_\Gamma^{-1}.
\end{aligned}$$

Recall $a = |\xi(\tilde{e}_i)|$, the lemma follows. $\square$

4 Positive entropy and regularity

So far we have only discussed algebraic properties of the group action, now we bring ergodic theory into the scene.

Notation 4.1. *By abuse of notation let* m *be the Lebesgue measure on* $\mathbb{R}^d$ *as well as that on* $\mathbb{T}^d$*. Denote by* m_X *the pushforward of* m *from* $\mathbb{R}^d$ *to* $\mathbb{R}^d \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ *by* ψ*. Abusing notation again, we call the projection of* m_X *to* X *by* m_X *as well. Notice as the Lebesgue measure on* $\mathbb{T}^d$ *is* G*-invariant,* m_X *is* $\times_{\phi(G)}$*-invariant. And on* X*,* m_X *is the pushforward of the Lebesgue measure* m *on* $\mathbb{T}^d$ *by* ψ*, thus the unique translation invariant probability measure.*

Let π *denote the projection from* $\mathbb{R}^d$ *to* $X = \mathbb{R}^d/\Gamma$*.*

As $|\det(\psi)| = \mathrm{Vol}(\psi(\mathbb{Z}^d)) = \mathrm{Vol}(\Gamma)$, $\mathrm{m}_X = \frac{\mathrm{Vol}}{\mathrm{Vol}(\Gamma)} = \mathcal{S}_\Gamma^{-d} \cdot \mathrm{Vol}$.

From now on let μ be a Borel probability measure on $\mathbb{T}^d$ satisfying Condition 1.8, where ϵ will be supposed later to be very small depending on α. We will always write

$$\eta = \mathcal{M}_\psi^{-1}\epsilon. \tag{4.1}$$

We define the pushforward measure $\tau = \psi_*\mu$ on X. Then it has similar properties.

Lemma 4.2. *Assuming Condition 1.8, then the inequality* $H_\tau(\mathcal{P}) \geq \alpha d \log \frac{1}{\eta} - d \log \mathcal{M}_\psi$ *holds for all measurable partitions* $\mathcal{P}$ *of* X *such that* $\mathrm{diam}\mathcal{P} \leq \eta \mathcal{S}_\Gamma$*.*

Proof. If the diameter of a partition $\mathcal{P}$ on X is at most $\eta\mathcal{S}_\Gamma$, then that of its pullback $\psi^*\mathcal{P}$ on $\mathbb{T}^d$ is bounded by $\|\psi^{-1}\|\eta\mathcal{S}_\Gamma \leq \mathcal{M}_\psi \eta = \epsilon$. By Condition 1.8, $H_\tau(\mathcal{P}) = H_\mu(\psi^*\mathcal{P}) \geq \alpha d \log \frac{1}{\epsilon} = \alpha d(\log \frac{1}{\eta} - \log \mathcal{M}_\psi) \geq \alpha \log \frac{1}{\eta} - d \log \mathcal{M}_\psi$ $\square$

4.1 Some non-conventional entropies

We demonstrated in §3.2 the irrationality of subspace V_i in X, a fact that makes it hard to construct measurable partitions on X that are consistent with both the projection π and the G-action simultaneously. For this reason we need some more technical arrangements.

Definition 4.3. *Let* μ *be a Borel probability measure on* X*. If a Borel set* $B \subset \mathbb{R}^d$ *is projected injectively by* π *into* X*, the* **homogeneous entropy** *of* B *with respect to* μ *is*

$$\overline{H}_\mu(B) = \frac{1}{\mathrm{m}_X(B)} \int_{x\in X} -\mu(x+B)\log\mu(x+B)\mathrm{dm}_X(x).$$

When $\mathcal{Q}$ *is a finite measurable partition of* B*, the* **homogeneous conditional entropy** *of* $\mathcal{Q}$ *with respect to* μ *is defined to be*

$$\overline{H}_\mu(B,\mathcal{Q}) = \frac{1}{\mathrm{m}_X(B)} \int_{x\in X} \sum_{Q\in\mathcal{Q}} -\mu(x+Q)\log\frac{\mu(x+Q)}{\mu(x+B)}\mathrm{dm}_X(x).$$

By change of variable, the following fact is immediate.

Lemma 4.4. *The homogeneous entropy and the homogeneous conditional entropy are translation invariant. Namely,* $\forall B, \mathcal{Q}$*,* $\forall y \in \mathbb{R}^d$*, let* $y + B$ *be the translation of* B *by the vector* y*, and* $y + \mathcal{Q} = \{y + Q | Q \in \mathcal{Q}\}$ *be the partition defined by translating* $\mathcal{Q}$ *by* y*. Then*

$$\overline{H}_\mu(y+B) = \overline{H}_\mu(B), \overline{H}_\mu(y+B, y+\mathcal{Q}) = \overline{H}_\mu(B,\mathcal{Q}).$$

Lemma 4.5. *Denote* $\mathrm{dm}_{\mu,B}(x) = \frac{1}{\mathrm{m}_X(B)}\mu(x+B)\mathrm{dm}_X(x)$. *Then* $\mathrm{m}_{\mu,B}$ *is a probability measure.*

Proof. The total mass is

$$\begin{aligned}\int_{x\in X}\mathrm{dm}_{\mu,B}(x) =&\frac{1}{\mathrm{m}_X(B)}\int_{x\in X}\mu(x+B)\mathrm{dm}_X(x)\\ =&\frac{1}{\mathrm{m}_X(B)}\int_{x\in X}\int_{y\in X}\mathbf{1}_{y-x\in B}\mathrm{d}\mu(y)\mathrm{dm}_X(x)\\ =&\frac{1}{\mathrm{m}_X(B)}\int_{y\in X}\int_{x\in X}\mathbf{1}_{y-x\in B}\mathrm{dm}_X(x)\mathrm{d}\mu(y)\\ =&\frac{1}{\mathrm{m}_X(B)}\int_{y\in X}\mathrm{m}_X(y-B)\mathrm{d}\mu(y)\\ =&\frac{1}{\mathrm{m}_X(B)}\int_{y\in X}\mathrm{m}_X(B)\mathrm{d}\mu(y)=1.\end{aligned}$$

□

Remark 4.6. $\overline{H}_\mu(B,\mathcal{Q})$ *can be written as*

$$\begin{aligned}&\frac{1}{\mathrm{m}_X(B)}\int_{x\in X}\mu(x+B)\sum_{Q\in\mathcal{Q}}(-\frac{\mu(x+Q)}{\mu(x+B)}\log\frac{\mu(x+Q)}{\mu(x+B)})\mathrm{dm}_X(x)\\ =&\mathbb{E}_{\mathrm{m}_{\mu,B}(x)}H_{\frac{\mu|_{x+B}}{\mu(x+B)}}(x+\mathcal{Q}).\end{aligned}$$

Where $H_{\frac{\mu|_{x+B}}{\mu(x+B)}}(x+\mathcal{Q})$ *stands for the usual measure theoretical entropy of the partition* $x+\mathcal{Q}$ *with respect to the renormalized probability measure* $\frac{\mu|_{x+B}}{\mu(x+B)}$.

In particular, we deduce

Corollary 4.7. *(i). For all* B,$\mathcal{Q}$ *and* μ, $\overline{H}_\mu(B,\mathcal{Q})\le\log|\mathcal{Q}|$.

(ii). If $\mathcal{Q}$ *and* $\mathcal{Q}'$ *are two different partitions of* B *then* $\overline{H}_\mu(B,\mathcal{Q}\vee\mathcal{Q}')\le\overline{H}_\mu(B,\mathcal{Q})+\overline{H}_\mu(B,\mathcal{Q}')$.

.

Definition 4.8. *Let* μ *be as above. The entropy of a finite measuable cover* $\mathcal{C}$ *of* X *is* $H_\mu(\mathcal{C}) = \sum_{C\in\mathcal{C}}-\mu(C)\log\mu(C)$.

We are going to cope with special B's and $\mathcal{Q}$'s.

For $\mathbf{v} = (v_i)_{i=1}^{r_1+r_2}, \mathbf{w} = (w_i)_{i=1}^{r_1+r_2} \in (\mathbb{R}_+\cup\{0\})^{\{1,\cdots,r_1+r_2\}}$, $\mathbf{v} > \mathbf{w}$ means $v_i > w_i, \forall i$. Let $\mathbf{1}_J = \mathbf{1}_{i\in J}\in(\mathbb{R}_+\cup\{0\})^{\{1,\cdots,r_1+r_2\}}$ be the charateristic vector of $J\subset\{1,\cdots,r_1+r_2\}$ and simply denote $\mathbf{1}=\mathbf{1}_{\{1,\cdots,r_1+r_2\}}$, $\mathbf{1}_i=\mathbf{1}_{\{i\}}$.

For $\mathbf{w}\in(\mathbb{R}_+\cup\{0\})^{\{1,\cdots,r_1+r_2\}}$, we define a family of boxes whose sides in V_i are of length 2^{-w_i}.

Definition 4.9. *For* $\mathbf{w}\in(\mathbb{R}_+\cup\{0\})^{\{1,\cdots,r_1+r_2\}}$, *a set* $B\subset\mathbb{R}^d=\oplus_{i=1}^{r_1+r_2}V_i$ *is a* $\mathbf{w}$*-box if* $B=\prod_{i=1}^{r_1+r_2}B_i$ *where:*

$B_i=[0,2^{-w_i}\mathcal{S}_\Gamma)$ *inside* V_i, *if* $V_i\cong\mathbb{R}$;

$B_i=\{x+\mathrm{i}y|x,y\in[0,2^{-w_i}\mathcal{S}_\Gamma)\}\subset V_i$, *i.e. the square* $[0,2^{-w_i}\mathcal{S}_\Gamma)\times[0,2^{-w_i}\mathcal{S}_\Gamma)$, *if* $V_i\cong\mathbb{C}$.

The family of all $\mathbf{w}$*-boxes are denoted by* $\Pi^{\mathbf{w}}$.

Definition 4.10. *For* $\mathbf{t} \in (\mathbb{N} \cup \{0\})^{\{1,\cdots,r_1+r_2\}}$.

Given $B \in \Pi^{\mathbf{w}}$, *set* $F_{\mathbf{t}}B = f_{\mathbf{t}}(B)$ *where* $f_{\mathbf{t}} = \oplus_{i=1}^{r_1+r_2} 2^{-t_i} \mathrm{Id}|_{V_i}$ *rescales each* V_i *by* 2^{-t_i}.

Then $F_{\mathbf{t}}B \in \Pi^{\mathbf{w}+\mathbf{t}}$ *is a subset of* B *and there is a unique partition of* B *into* $2^{\sum_{i=1}^{r_1+r_2} d_i \cdot t_i}$ *different translates of* $F_{\mathbf{t}}(B)$. *Call this partition by* $\mathcal{Q}_{\mathbf{t}}B$.

$F_{\mathbf{t}}$ maps $\Pi^{\mathbf{w}}$ to $\Pi^{\mathbf{w}+\mathbf{t}}$. It is an immediate observation that

$$F_{\mathbf{t}} \circ F_{\mathbf{s}} = F_{\mathbf{t}+\mathbf{s}}. \tag{4.2}$$

Lemma 4.11. *Any box in* $\Pi^{\mathbf{w}}$ *is projected injectively by* π *if* $\mathbf{w} \geq \log(\sqrt{d}\mathcal{M}_\psi)\mathbf{1}$.

Proof. If $B \in \Pi^{\mathbf{w}}$, then all side of it are at most of length $2^{-\log(\sqrt{d}\mathcal{M}_\psi)}\mathcal{S}_\Gamma$. So $\forall x, x' \in B$, $|x - x'| < \sqrt{d} \cdot 2^{-\log(\sqrt{d}\mathcal{M}_\psi)}\mathcal{S}_\Gamma = \mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma$. Hence $|\psi^{-1}(x - x')| \leq \|\psi^{-1}\|\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma \leq 1$ by Definition 2.5. Thus unless $x = x'$, $\psi^{-1}(x - x') \notin \mathbb{Z}^d$, or equivalently, $x - x' \notin \psi(\mathbb{Z}^d) = \Gamma$ and x, x' project to different points in $X = \mathbb{R}^d/\Gamma$. □

Lemma 4.12. *If* $\pi|_B$ *is injective,* μ *is a Borel probability measure on* X *and* $\mathbf{t}^1, \cdots, \mathbf{t}^n \in (\mathbb{N} \cup \{0\})^{\{1,\cdots,r_1+r_2\}}$, *then*

$$\overline{H}_\mu(F_{\sum_{s=1}^n \mathbf{t}^s}B) = \overline{H}_\mu(B) + \sum_{s=1}^n \overline{H}_\mu(F_{\sum_{l=1}^{s-1}\mathbf{t}^l}B, \mathcal{Q}_{\mathbf{t}^s}F_{\sum_{l=1}^{s-1}\mathbf{t}^l}B).$$

Proof. The statement is trivial for $n = 0$ by (4.2), to conclude by induction it suffices to do the $n = 1$ case. In this case there is only one vector $\mathbf{t} = \mathbf{t}^1$. By Lemma 4.4, $\overline{H}_\mu(F_{\mathbf{t}}B) = H_\mu(Q), \forall Q \in \mathcal{Q}_{\mathbf{t}}B$, therefore

$$\begin{aligned}
\overline{H}_\mu(F_{\mathbf{t}}B) =& \frac{1}{|\mathcal{Q}_{\mathbf{t}}B|} \sum_{Q \in \mathcal{Q}_{\mathbf{t}}B} \overline{H}_\mu(Q) \\
=& \frac{1}{|\mathcal{Q}_{\mathbf{t}}B|} \sum_{Q \in \mathcal{Q}_{\mathbf{t}}B} \frac{|\mathcal{Q}_{\mathbf{t}}B|}{\mathrm{m}_X(B)} \int_{x \in X} -\mu(x+Q)\log\mu(x+Q)\mathrm{dm}_X(x) \\
=& \frac{1}{\mathrm{m}_X(B)} \int_{x \in X} \sum_{Q \in \mathcal{Q}_{\mathbf{t}}B} -\mu(x+Q)\log\mu(x+Q)\mathrm{dm}_X(x).
\end{aligned}$$

By decomposing $-\mu(x+Q)\log\mu(x+Q)$ into the sum of $-\mu(x+Q)\log\mu(x+B)$ and $-\mu(x+Q)\log\frac{\mu(x+Q)}{\mu(x+B)}$, we obtain

$$\begin{aligned}
&\overline{H}_\mu(F_{\mathbf{t}}B) \\
=& \frac{1}{\mathrm{m}_X(B)} \int_{x \in X} -\mu(x+B)\log\mu(x+B)\mathrm{dm}_X(x) \\
&+ \frac{1}{\mathrm{m}_X(B)} \int_{x \in X} \sum_{Q \in \mathcal{Q}_{\mathbf{t}}B} -\mu(x+Q)\log\frac{\mu(x+Q)}{\mu(x+B)}\mathrm{dm}_X(x) \\
=& \overline{H}_\mu(B) + \overline{H}_\mu(B, \mathcal{Q}_{\mathbf{t}}B).
\end{aligned}$$

This establishes the lemma. □

4.2 Positive entropy in an eigenspace

In this part we will show that when both Conditions 1.8 and 2.8 are satisfied, there exists an index i such that the measure $\tau = \psi_*\mu$ has positive entropy in the direction of V_i with respect to some scale that we will specify.

Let us start with a box $B_0 \in \Pi^{R_0\mathbf{1}}$ where

$$R_0 := \log \frac{\sqrt{d}}{\eta} = \log \frac{\sqrt{d}\mathcal{M}_\psi}{\epsilon}, \tag{4.3}$$

then $\operatorname{diam}(B_0) = \sqrt{d}2^{-R_0}\mathcal{S}_\Gamma = \eta\mathcal{S}_\Gamma$. As $\epsilon < 1$, $R_0 \geq \log(\sqrt{d}\mathcal{M}_\psi)$ always holds and $\pi|_{B_0}$ is injective by Lemma 4.11. We now try to give an estimate for the average entropy $\overline{H}_\tau(B_0)$.

Lemma 4.13. *Assuming Conditions 1.8 and 2.8, if a constant $\kappa_4 = \kappa_4(d)$ is sufficiently large then for any $\delta \geq \frac{\kappa_4\mathcal{M}_\psi^d}{R_0}$ and box $B_0 \in \Pi^{R_0\mathbf{1}}$,*

$$\overline{H}_\tau(B_0) \geq (\alpha - \delta)dR_0,$$

where $\tau = \psi_*\mu$ with ψ and μ coming respectively from Conditions 1.8 and 2.8

Proof. Let $b_i = \psi(e_i)$, then $|b_i| \leq \|\psi\| \leq \mathcal{M}_\psi\mathcal{S}_\Gamma$. The parallelepiped $D = \{\sum_{j=1}^d \theta_j b_j | \theta_j \in [0,1), \forall j\}$ spanned by the linear basis $\{b_1, \cdots, b_d\}$ for the lattice Γ is a fundamental domain of the projection π.

The preimage $\psi^{-1}(B_0) \subset \mathbb{R}^d$ is inside a ball of radius $\|\psi^{-1}\|\operatorname{diam}(B_0) \leq \mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}\cdot\eta\mathcal{S}_\Gamma = \mathcal{M}_\psi\eta = \epsilon$. Hence $\psi^{-1}(B_0 - B_0) = \psi^{-1}(B_0) - \psi^{-1}(B_0)$ is covered by the ball of radius 2ϵ centered at the origin. Notice $\psi^{-1}(D) = [0,1)^d \subset \mathbb{R}^d$, hence

$$\psi^{-1}(D + B_0 - B_0) \subset (-2\epsilon, 1 + 2\epsilon)^d, \tag{4.4}$$

which can be covered by at most 5^d translates of $\psi^{-1}(D)$ as $\epsilon < 1$. Thus $D + B_0 - B - 0$ is covered by at most 5^d translates of D, each of which is a fundamental domain of π. Hence $\forall x \in X$, $|\pi^{-1}(x) \cap (D + B_0 - B_0)| \leq 5^d$.

We may tile $\mathbb{R}^d$ by translating B_0: $\mathbb{R}^d = \sqcup_{y\in\Sigma}(y + B_0)$ where $\Sigma \subset \mathbb{R}^d$ is a lattice. Denote $Y = \{y \in \Sigma | (y + B_0) \cap D \neq \emptyset\}$. Then $D \subset D_+ \subset D + B_0 - B_0$ where $D_+ = \sqcup_{y\in Y}(y + B_0)$.

Let $\mathcal{C}$ be the collection $\{\pi(y + B_0)|y \in Y\}$ then it is a cover of X as $\cup_{C\in\mathcal{C}}C \supset \cup_{y\in Y}\pi(y + B_0) = \pi(D_+) \supset \pi(D) = X$. Furthermore every point in X is covered by at most 5^d pieces from $\mathcal{C}$ as $|\{C \in \mathcal{C}|x \in C\}| = |\{y \in Y|x \in \pi(y + B_0)\}| = |\pi^{-1}(x) \cap D_+| \leq |\pi^{-1}(x) \cap (D + B_0 - B_0)| \leq 5^d$.

Denote $x + \mathcal{C} = \{x + C|C \in \mathcal{C}\}$ for all $x \in X$. $x + \mathcal{C}$ is the translate of $\mathcal{C}$ by x and hence remains a cover of X whose multiplicity at any given point is at most 5^d. Then

$$\begin{aligned}
&\int_{x\in X} H_\tau(x + \mathcal{C})\mathrm{dm}_X(x) \\
=&\int_{x\in X} \sum_{C\in\mathcal{C}} -\tau(x + C)\log\tau(x + C)\mathrm{dm}_X(x) \\
=&\int_{x\in X} \sum_{y\in Y} -\tau(x + y + B_0)\log\tau(x + y + B_0)\mathrm{dm}_X(x).
\end{aligned} \tag{4.5}$$

As $\mathrm{m}_X(y+B_0)=\mathrm{m}_X(B_0)$ and $\overline{H}_\tau(y+B_0)=\overline{H}_\tau(B_0)$ for all $y\in Y$, by definition of $\overline{H}_\tau(y+B_0)$,

$$\begin{aligned}(4.5)=&\sum_{y\in Y}\mathrm{m}_X(B_0)\overline{H}_\tau(y+B_0)=|\mathcal{C}|\mathrm{m}_X(B_0)\overline{H}_\tau(B_0)\\ =&\mathrm{m}_X(D_+)\overline{H}_\tau(B_0),\end{aligned}\tag{4.6}$$

The cover $\mathcal{C}$ induces a partition $\mathcal{P}_\mathcal{C}$ of X:

$$\mathcal{P}_\mathcal{C}=\{\pi\big((y+B_0)\cap D\big)|y\in Y\}.$$

Again let $x+\mathcal{P}_\mathcal{C}=\{x+P|P\in\mathcal{P}_\mathcal{C}\}$ denote its translate by x. There is a one-to-one correspondence between $\mathcal{C}$ and $\mathcal{P}_\mathcal{C}$: $P_C=\pi\big((y+B_0)\cap D\big)\leftrightarrow C=\pi(y+B_0)$ and for all C we have $P_C\subset C$. As $C\in\mathcal{C}$ is always a translate of B_0 we know the diameter of $\mathcal{P}_\mathcal{C}$ is at most $\eta\mathcal{S}_\Gamma$ and so is that of $x+\mathcal{P}_\mathcal{C}$, $\forall x\in X$. So by Lemma 4.2,

$$H_\tau(x+\mathcal{P}_\mathcal{C})\geq\alpha d\log\frac{1}{\eta}-d\mathcal{M}_\psi.\tag{4.7}$$

On the other hand, notice

$$\begin{aligned}&H_\tau(x+\mathcal{C})\\ =&\sum_{C\in\mathcal{C}}-\tau(x+C)\log\tau(x+C)\\ \geq&\sum_{C\in\mathcal{C}}-\tau(x+P_C)\log\tau(x+C)\\ =&\sum_{C\in\mathcal{C}}\big(-\tau(x+P_C)\log\tau(x+P_C)+\tau(x+P_C)\log\frac{\tau(x+P_C)}{\tau(x+C)}\big)\\ =&H_\tau(x+\mathcal{P}_\mathcal{C})-\sum_{C\in\mathcal{C}}\tau(x+C)\big(-\frac{\tau(x+P_C)}{\tau(x+C)}\log\frac{\tau(x+P_C)}{\tau(x+C)}\big).\end{aligned}\tag{4.8}$$

It follows from (4.7), (4.8), the inequality $-u\log u\leq\frac{1}{e\ln 2},\forall u\in[0,1]$ and the fact that the multiplicity of the cover $x+\mathcal{C}$ is at most 5^d that

$$\begin{aligned}H_\tau(x+\mathcal{C})\geq&\alpha d\log\frac{1}{\eta}-d\log\mathcal{M}_\psi-\frac{1}{e\ln 2}\sum_{C\in\mathcal{C}}\tau(x+C)\\ \geq&\alpha dR_0-\alpha d\log\sqrt{d}-d\log\mathcal{M}_\psi-\frac{1}{e\ln 2}\cdot 5^d,\end{aligned}\tag{4.9}$$

which, together with (4.6), implies

$$\overline{H}_\tau(B_0)\geq\frac{1}{\mathrm{m}_X(D_+)}\big(\alpha dR_0-O_d(\log\mathcal{M}_\psi)-O_d(1)\big).\tag{4.10}$$

By (4.4), the volume of $\psi^{-1}\big((D+B_0-B_0)\backslash D\big)$ is bounded by $(1+4\epsilon)^d-1$, hence

$$\begin{aligned}&\mathrm{m}_X(D_+)-1\\ \leq&\mathrm{m}_X(D+B_0-B_0)-\mathrm{m}_X(D)=\mathrm{m}_X\big((D+B_0-B_0)\backslash D\big)\\ =&\mathrm{m}\Big(\psi^{-1}\big((D+B_0-B_0)\backslash D\big)\Big)\\ \leq&(1+4\epsilon)^d-1=O_d(\epsilon),\end{aligned}\tag{4.11}$$

where the implied dimensional constant can be made explicit.

From (4.10) and (4.11) we conclude

$$\overline{H}_\tau(B_0) \geq \frac{1}{1+O_d(\epsilon)}\big(\alpha d R_0 - O_d(\log \mathcal{M}_\psi) - O_d(1)\big), \tag{4.12}$$

with effective implied constants.

So when $\delta \in [\frac{\kappa_4 \mathcal{M}_\psi^d}{R_0}, \frac{\alpha}{10}]$, as $\mathcal{M}_\psi \geq 1$,

$$\epsilon = \frac{1}{\sqrt{d}\mathcal{M}_\psi} 2^{-R_0} \lesssim_d 2^{-R_0} \lesssim \frac{1}{R_0} \leq \frac{\delta}{\kappa_4} < \frac{\delta}{\kappa_4 \alpha};$$

in addition

$$O_d(\log \mathcal{M}_\psi) + O_d(1) = O_d(\mathcal{M}_\psi^d) = O_d(\frac{\delta R_0}{\kappa_4}).$$

So by (4.12),

$$\overline{H}_\tau(B_0) \geq \frac{1}{1+O_d(\frac{\delta}{\kappa_4\alpha})}\big(\alpha d R_0 - O_d(\frac{\delta R_0}{\kappa_4})\big) \geq (\alpha-\delta) d R_0,$$

if κ_4 is sufficiently large (depending on d). □

The following fact is a consequence to additivity (Lemma 4.12)

Lemma 4.14. *For a sufficiently large effective constant $\kappa_4 = \kappa_4(d)$, suppose $\delta \geq \frac{\kappa_4 \mathcal{M}_\psi^d}{R_0}$ and $T \leq \frac{\delta R_0}{2}$ be an integer. Then there exist a real number $R \in [\delta R_0, R_0 - T]$ and a box $B \in \Pi^{R\mathbf{1}}$ such that*

$$\overline{H}_\tau(B, \mathcal{Q}_{T\mathbf{1}} B) \geq (\alpha - 3\delta) d T.$$

Proof. Write $R_0 = S + pT$ where $S \in [\delta R_0, \delta R_0 + T)$ and $p \in \mathbb{N}$. Note $S \geq \kappa_4 \mathcal{M}_\psi^d$ by the assumption about δ. We can always take $\kappa_4 = \kappa_4(d)$ large to make sure $\kappa_4 \mathcal{M}_\psi^d \geq \log(\sqrt{d}\mathcal{M}_\psi)$ for all $\mathcal{M}_\psi \geq 1$. Hence there is $S_- \in [\log(\sqrt{d}\mathcal{M}_\psi), \log(\sqrt{d}\mathcal{M}_\psi)+1]$ such that $S - S_- = q \in \mathbb{N} \cup \{0\}$.

Fix $B_0 \in \Pi^{R_0\mathbf{1}}$, $B_{S_-} \in \Pi^{S_-\mathbf{1}}$ and $B_S \in \Pi^{S\mathbf{1}}$ such that $B_0 = F_{pT\mathbf{1}} B_S$ and $B_S = F_{q\mathbf{1}} B_{S_-}$. By Lemma 4.12 and 4.13

$$\begin{aligned} &\overline{H}_\tau(B_{S_-}) + \overline{H}_\tau(B_{S_-}, \mathcal{Q}_{q\mathbf{1}} B_{S_-}) + \sum_{r=0}^{p-1} \overline{H}_\tau(F_{rT\mathbf{1}} B_S, \mathcal{Q}_{T\mathbf{1}} F_{rT\mathbf{1}} B_S) \\ =&\overline{H}_\tau(B_0) \geq (\alpha - \delta) d T. \end{aligned} \tag{4.13}$$

On the other hand, using again $-u \log u \leq \frac{1}{e \ln 2}$,

$$\begin{aligned} \overline{H}_\tau(B_{S_-}) =& \frac{1}{\mathrm{m}_X(B_{S_-})} \int_{x \in X} -\nu(x + B_{S_-}) \log \nu(x + B_{S_-}) \mathrm{dm}_X(x) \\ \leq& \frac{(2\sqrt{d})^d \mathcal{M}_\psi^d}{e \ln 2}, \end{aligned} \tag{4.14}$$

because $\mathrm{m}_X(B_{S_-}) = \mathcal{S}_\Gamma^{-d} \mathrm{vol}(B_{S_-}) = \mathcal{S}_\Gamma^{-d}\big(2^{-S_-}\mathcal{S}_\Gamma\big)^d = 2^{-dS_-}$ has lower bound $2^{-d\left(\log(\sqrt{d}\mathcal{M}_\psi)+1\right)} = (2\sqrt{d})^{-d}\mathcal{M}_\psi^{-d}$.

Moreover by Corollary 4.7

$$\begin{aligned}\overline{H}_\tau(B_{S_-}, \mathcal{Q}_{q\mathbf{1}}B_{S_-}) \leq &\log|\mathcal{Q}_{q\mathbf{1}}B_{S_-}| \leq dq \leq dS \leq d(\delta R_0 + T)\\ \leq &d(\delta R_0 + \frac{\delta R_0}{2}).\end{aligned} \tag{4.15}$$

Notice by taking the explicit constant $\kappa_4(d)$ large enough we have $d\delta R_0 \geq d\kappa_4 \mathcal{M}_\psi^d > 2\cdot\frac{(2\sqrt{d})^d\mathcal{M}_\psi^d}{e\ln 2}$. So by (4.13), (4.14) and (4.15), $\exists r \in \{0, 1, \cdots, p-1\}$ such that

$$\begin{aligned}&\overline{H}_\tau(F_{rT\mathbf{1}}B_S, \mathcal{Q}_{T\mathbf{1}}F_{rT\mathbf{1}}B_S)\\ \geq &\frac{1}{p}\big((\alpha-\delta)dR_0 - d(\delta R_0 + \frac{\delta R_0}{2}) - \frac{(2\sqrt{d})^d\mathcal{M}_\psi^d}{e\ln 2}\big)\\ \geq &\frac{1}{p}\big((\alpha-\delta)dR_0 - d(\delta R_0 + \frac{\delta R_0}{2}) - \frac{d\delta R_0}{2}\big)\\ = &(\alpha - 3\delta)d\frac{R_0}{p}\\ \geq &(\alpha-3\delta)dT\end{aligned} \tag{4.16}$$

$F_{rT\mathbf{1}}B_S \in \Pi^{R\mathbf{1}}$ where $R = S + rT$. As $R \geq S \geq \delta R_0$ and $R \leq S + (p-1)T = R_0 - T$, the lemma follows. $\square$

The lemma basically claims τ has positive entropy at scale $2^{-R}\mathcal{S}_\Gamma$. It follows from subadditivity (Corollary 4.7) that the projection of τ to at least one of the V_i's has positive entropy at the same scale, which is characterized by:

Corollary 4.15. *There is an index $i \in \{1, \cdots, r_1 + r_2\}$ such that*

$$\overline{H}_\tau(B, \mathcal{Q}_{T\mathbf{1}_i}B) \geq (\alpha - 3\delta)d_iT.$$

Proof. Since $\mathcal{Q}_{T\mathbf{1}}B = \bigvee_{i=1}^{r_1+r_2}\mathcal{Q}_{T\mathbf{1}_i}B$, by Corollary 4.7

$$\sum_{i=1}^{r_1+r_2}\overline{H}_\tau(B, \mathcal{Q}_{T\mathbf{1}_i}B) \geq \overline{H}_\tau(B, \mathcal{Q}_{T\mathbf{1}}B) \geq (\alpha-3\delta)dT.$$

Recall $d_i = \dim V_i$; as $\sum_{i=1}^{r_1+r_2} d_i = d$, the corollary is proved. $\square$

4.3 From entropy to L^2-norm

Using the positive entropy of τ in V_i direction, we now construct a new measure dominated by τ so that its total mass is bounded from below but a certain L^2-norm in V_i direction is bounded from above.

Let T and B be as in Lemma 4.14. For all $x \in X$ such that $\tau(x + B) \neq 0$, define a probability measure $\tau_x = \frac{\tau|_{x+B}}{\tau(x+B)}$, supported on $x + B$. Set $\mathcal{S}_x = \{Q \in \mathcal{Q}_{T\mathbf{1}_i}B | \frac{\tau(x+Q)}{\tau(x+B)} \leq 2^{-d_i\delta T}\} = \{Q \in \mathcal{Q}_{T\mathbf{1}_i}B | \tau_x(x+Q) \leq 2^{-d_i\delta T}\}$ and $\mathcal{S}'_x = \mathcal{Q}\backslash\mathcal{S}_x$.

Notice τ_x, $\mathcal{S}_x$ and $\mathcal{S}'_x$ are defined for $\mathrm{m}_{\tau,B}$-a.e. x, where $\mathrm{m}_{\tau,B}$ is defined as in Lemma 4.5. Quoting Remark 4.6 and Lemma 4.14 we have

$$
\begin{aligned}
&(\alpha-3\delta)d_iT\\
\le&\overline{H}_\tau(B,\mathcal{Q}_{T\mathbf{1}_i}B)\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}H_{\tau_x}(x+\mathcal{Q}_{T\mathbf{1}_i}B)\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}H_{\tau_x}\Big(\{\bigcup_{Q\in\mathcal{S}_x}(x+Q),\bigcup_{Q\in\mathcal{S}'_x}(x+Q)\}\Big)\\
&\quad+\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}H_{\tau_x}\Big(x+\mathcal{Q}_{T\mathbf{1}_i}B\big|\{\bigcup_{Q\in\mathcal{S}_x}(x+Q),\bigcup_{Q\in\mathcal{S}'_x}(x+Q)\}\Big)\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}H_{\tau_x}\Big(\{\bigcup_{Q\in\mathcal{S}_x}(x+Q),\bigcup_{Q\in\mathcal{S}'_x}(x+Q)\}\Big)\\
&\quad+\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{Q\in\mathcal{S}_x}\tau_x(x+Q)\cdot\big(-\log\frac{\tau_x(x+Q)}{\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))}\big)\\
&\quad\quad+\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{Q\in\mathcal{S}'_x}\tau_x(x+Q)\cdot\big(-\log\frac{\tau_x(x+Q)}{\tau_x(\bigcup_{Q\in\mathcal{S}'_x}(x+Q))}\big)
\end{aligned}
\tag{4.17}
$$

Observe all three terms on the right-hand side can be bounded from above:
By Corollary 4.7, $H_{\tau_x}\Big(\{\bigcup_{Q\in\mathcal{S}_x}(x+Q),\bigcup_{Q\in\mathcal{S}'_x}(x+Q)\}\Big)\le\ln 2$;
Moreover,

$$
\begin{aligned}
&\sum_{Q\in\mathcal{S}_x}\tau_x(x+Q)\cdot\big(-\log\frac{\tau_x(x+Q)}{\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))}\big)\\
=&\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))H_{\frac{\tau_x|_{\bigcup_{Q\in\mathcal{S}_x}(x+Q)}}{\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))}}(\mathcal{S}_x)\\
\le&\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))\cdot\log|\mathcal{S}_x|\\
\le&\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))\cdot\log|\mathcal{Q}_{T\mathbf{1}_i}B|\\
=&\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))\cdot\log 2^{d_iT}\\
=&d_iT\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q)).
\end{aligned}
$$

Last,

$$
\begin{aligned}
&\sum_{Q\in\mathcal{S}'_x} \tau_x(x+Q)\cdot\big(-\log\frac{\tau_x(x+Q)}{\tau_x(\bigcup_{Q\in\mathcal{S}'_x}(x+Q))}\big)\\
\le&\sum_{Q\in\mathcal{S}'_x} \tau_x(x+Q)\cdot\big(-\log\tau_x(x+Q)\big)\\
\le&\sum_{Q\in\mathcal{S}'_x} \tau_x(x+Q)\cdot(-\log 2^{-d_i\delta T})\\
=&d_i\delta T\sum_{Q\in\mathcal{S}'_x} \tau_x(x+Q)\\
=&d_i\delta T
\end{aligned}
$$

So (4.17) gives

$$
(\alpha-3\delta)d_iT\le \ln 2+d_iT\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))+d_i\delta T. \tag{4.18}
$$

In consequence, if

$$
\delta T\ge 1 \tag{4.19}
$$

then $d_i\delta T\ge\delta T\ge\ln 2$ and by (4.18), $d_iT\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))\ge(\alpha-3\delta)d_iT-\ln 2-d_i\delta T\ge(\alpha-5\delta)d_iT$, so we get

$$
\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\tau_x(\bigcup_{Q\in\mathcal{S}_x}(x+Q))\ge\alpha-5\delta. \tag{4.20}
$$

For all x such that $\tau(x+B)\ne 0$ set

$$
\nu_x:=\sum_{Q\in\mathcal{S}_x}\tau_x|_{x+Q},
$$

which is supported on $\bigcup_{Q\in\mathcal{S}'_x}(x+Q)\subset x+B$. Then

$$
|\nu_x|\le 1 \tag{4.21}
$$

as ν_x is bounded by the probability measure τ_x.

If We define a new measure

$$
\nu:=\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\nu_x,
$$

then it follows from (4.20) that the total mass

$$
|\nu|\ge\alpha-5\delta. \tag{4.22}
$$

Moreover, observe

$$
\begin{aligned}
\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\tau_x=&\frac{1}{\mathrm{m}_X(B)}\int_{x\in X}\frac{\tau|_{x+B}}{\tau(x+B)}\cdot\tau(x+B)\mathrm{dm}_X(x)\\
=&\frac{1}{\mathrm{m}_X(B)}\int_{x\in X}\tau\cdot\mathbf{1}_{x+B}\mathrm{dm}_X(x)\\
=&\frac{\tau}{\mathrm{m}_X(B)}\int_{x\in X}\mathbf{1}_{x+B}\mathrm{dm}_X(x)\\
=&\frac{\tau}{\mathrm{m}_X(B)}\cdot\mathrm{m}_X(B)\mathbf{1}_X=\tau.
\end{aligned}
$$

Hence

$$\nu = \mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\nu_x \leq \mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\tau_x = \tau. \tag{4.23}$$

Further, remark

$$\begin{aligned}
&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \sum_{Q\in\mathcal{Q}_{T\mathbf{1}_i}B} \nu_x^2(x+Q)\\
\leq&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\big(\max_{Q\in\mathcal{Q}_{T\mathbf{1}_i}B}\nu_x(x+Q)\big)\big(\sum_{Q\in\mathcal{Q}_{T\mathbf{1}_i}B}\nu_x(x+Q)\big)\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\big(\max_{Q\in\mathcal{S}_x}\nu_x(x+Q)\big)\nu_x(x+B),
\end{aligned} \tag{4.24}$$

where the last step was because ν_x is supported on $x+\cup_{Q\in\mathcal{S}_x}Q$. Since $\forall Q\in\mathcal{S}_x$, restricted to $x+Q$, ν_x is identical to τ_x,

$$\begin{aligned}
(4.24)\ =&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\big(\max_{Q\in\mathcal{S}_x}\tau_x(x+Q)\big)\nu_x(x+B)\\
\leq&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}2^{-d_i\delta T}\nu_x(x+B)\\
\leq&2^{-d_i\delta T}.
\end{aligned} \tag{4.25}$$

To sum up before finishing Section 4, what has been proved so far (4.21, 4.22, 4.23, 4.25) is the following proposition.

Proposition 4.16. *There is a constant κ_4 depending effectively on the dimension d such that if Conditions 2.8 and 1.8 are satisfied with $\delta\in[\frac{\kappa_4\mathcal{M}_\psi^d}{R_0},\frac{\alpha}{10}]$, then $\forall T\in[\frac{1}{\delta},\frac{\delta R_0}{2}]$, where $\sqrt{d}2^{-R_0}=\eta=\mathcal{M}_\psi^{-1}\epsilon$, there exist:*

(i). A number $R\in[\delta R_0, R_0-T]$;

(ii). A box $B\in\Pi^{R\mathbf{1}}$;

(iii). A probability measure $\mathrm{m}_{\tau,B}$ on X;

(iv). A family of measures ν_x supported on $x+B$ defined for $\mathrm{m}_{\tau,B}$-a.e. x;

(v). An index $i\in\{1,2,\cdots,r_1+r_2\}$;

such that the measure ν given by $\nu=\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\nu_x$ satisfies:

(1). $\nu\leq\tau$, where $\tau=\psi_*\mu$;

(2). $|\nu_x|\leq 1$, $\mathrm{m}_{\tau,B}$-a.e. $x\in X$;

(3). $|\nu|\geq\alpha-5\delta$;

(4). $\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{Q\in\mathcal{Q}_{T\mathbf{1}_i}B}\nu_x^2(x+Q)\leq 2^{-d_i\delta T}$, where $d_i=\dim V_i$.

5 Group action on a single eigenspace

In the special direction V_i which is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ we obtained from last section, the group G acts multiplicatively: $\forall v\in V_i, g\in G$, $\times_{\phi(g)}.v=\sigma_i(\phi(g))v=\zeta_g^i v$.

From now on let i be fixed. We are going to discuss several types of behavior of the G-action on V_i that can help us.

5.1 Expansion

This is the simplest behavior of the G-action that we study. When the j-th eigenvalue ζ_g^j of an element $g \in G$ is of absolute value greater than 1 (resp. less than 1), we say g *expands* (resp. *contracts*) V_i, which is equivalent to that $\mathcal{L}(\phi(g))$ lies in $\{w \in W | w_i > 0\}$ (resp. $\{w \in W | w_i < 0\}$), where $\mathcal{L}$ and W are defined as in §2.1.

Definition 5.1. *In an abelian group, two elements x, y are said to be* **multiplicatively independent** *if there doesn't exist $(p,q) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ such that $x^p y^q$ is identity.*

It follows easily from Dirichlet's Unit Theorem that there exists an element from G that expands V_i and contracts all $V_j, j \neq i$. Actually we can say more.

Proposition 5.2. *Given any $i \in \{1, \cdots, r_1 + r_2\}$, there is a pair of multiplicatively independent elements $u, \tilde{u}$ from G such that*

$$\begin{cases} h^{\mathrm{Mah}}(\phi(u)), h^{\mathrm{Mah}}(\phi(\tilde{u})) < 9d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}; \\ |\zeta_u^i| \geq 2^{d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}, |\zeta_u^j| < 1, \forall j \neq i; \\ |\zeta_{\tilde{u}}^i| \geq 2^{\frac{1}{4}d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}. \end{cases}$$

Here h is the logarithmic Mahler measure introduced in Definition 2.2.

Moreover, in case that $V_i \cong \mathbb{C}$ it can be required that $\zeta_{u^n}^i \notin \mathbb{R}, \forall n \in \mathbb{N}$ and if $\zeta_{u^p\tilde{u}^q}^i \in \mathbb{R}$ and $|\zeta_{u^p\tilde{u}^q}^i| > 1$ for some pair $(p,q) \in \mathbb{Z}^2$, then $|\zeta_{u^p\tilde{u}^q}^i| \geq 2^{\frac{1}{4}d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}$.

It should be pointed out that the proof of proposition could be much shorter if the last paragraph was dropped from statement. However this restriction is going to be an essential ingredient in obtaining effectiveness in §5.3.

Proof. Define a linear norm

$$h_0(w) = \frac{1}{2}\sum_{j=1}^{r_1+r_2} d_j|w_j| = \sum_{j=1}^{r_1+r_2} d_j \max(w_j, 0) = -\sum_{j=1}^{r_1+r_2} d_j \min(w_j, 0) \tag{5.1}$$

on W and let $\Omega = \{w \in W | h_0(w) \leq 1\}$. Observe $\forall w \in W, \forall j$,

$$|w_j| \leq h_0(w). \tag{5.2}$$

Moreover, for a unit $t \in U_K$, $h^{\mathrm{Mah}}(t) = h_0(\mathcal{L}(t))$.

We also define a positive definite quadratic form Q_0 on W by

$$Q_0(w,z) = \frac{1}{2}\sum_{j=1}^{r_1+r_2} d_j w_j z_j \tag{5.3}$$

.

If we denote $w_{r_1+r_2+j} = w_{r_1+j}$ for $1 \le j \le r_2$ then $\sum_{j=1}^d w_d = 0$, $Q_0(w,z) = \frac{1}{2}\sum_{j=1}^d w_j z_j$, $h_0(w) = \sum_{j=1}^d \max(w_j, 0)$. So

$$\begin{aligned} Q_0(w,w) =& \frac{1}{2}\sum_{j=1}^d w_j^2 \\ =& \frac{1}{2}\Big(\sum_{1\le j\le d, w_j>0} w_j^2 + \sum_{1\le j\le d, w_j<0} w_j^2\Big) \\ \le& \frac{1}{2}\big((\sum_{1\le j\le d, w_j>0} w_j)^2 + (\sum_{1\le j\le d, w_j<0} w_j)^2\big) \\ =& \frac{1}{2}\big(h_0^2(w) + h_0^2(w)\big) \\ =& h_0^2(w). \end{aligned} \tag{5.4}$$

And by Cauchy-Schwarz inequality $|Q_0(w,z)| \le h_0(w)h_0(z)$.

If $V_i \cong \mathbb{C}$ and $F := \sigma_i^{-1}(\mathbb{R})$ is not $\mathbb{Q}$, then by Lemma 2.7 there are $1 \le k, l \le r_1 + r_2$ such that k, l, i are distinct and for any unit t from U_F, $\mathcal{L}(t)$ is in the hyperplane

$$\forall t \in U_F,\ \mathcal{L}(t) \in W_{k,l} := \{w_k = w_l\}. \tag{5.5}$$

If $V_i \cong \mathbb{R}$ or $\sigma_i^{-1}(\mathbb{R}) = \mathbb{Q}$, then we fix any pair (k,l) so that k,l,i are distinct, which is possible as $r_1 + r_2 = r + 1 \ge 3$. So in case that $V_i \cong \mathbb{C}$ and $F := \sigma_i^{-1}(\mathbb{R}) = \mathbb{Q}$, as $U_F = \{\pm 1\}$, (5.5) still holds.

Construct $a, b \in W$ by

$$a_i = \frac{1}{d_i},\ a_j = -\frac{1}{d-d_i}, \forall j \ne i;$$
$$b_k = \frac{1}{d_k},\ b_k = -\frac{1}{d_l},\ b_j = 0, \forall j \ne k, l.$$

It is easy to compute $h_0(a) = h_0(b) = 1$, moreover

$$Q_a := Q_0(a,a) = \frac{1}{2}\Big(\frac{1}{d_i} + \frac{1}{d-d_i}\Big) \ge \frac{1}{4},\ Q_b := Q_0(b,b) = \frac{1}{2}\Big(\frac{1}{d_k} + \frac{1}{d_l}\Big) \ge \frac{1}{2}$$

and $Q_0(a,b) = 0$.

Notice the hyperplane $W_{k,l}$ is the orthogonal complement to b with respect to Q_0. So $\forall t \in U_F$, $Q_0(\mathcal{L}(t), b) = 0$ if $V_i \cong C$, $F = \sigma_i^{-1}(\mathbb{R})$.

$\phi(G)$, a finite-index subgroup of U_K, is embedded as a full-rank lattice in W by $\mathcal{L}$. Let $0 < m_1 \le m_2 \le \cdots \le m_r$ be the successive minima of $\mathcal{L}(\phi(G))$ with respect to Ω then by definition $m_r = \mathcal{F}_{\phi(G)}$.

By a theorem of Jarník on inhomogeneous minimum (cf. [9, p99]), for any point z in W, there is a lattice point w from $\mathcal{L}(\phi(G))$ such that $w - z \in m^*\Omega$ where $m^* = \frac{\sum_{j=1}^r m_i}{2} \le \frac{1}{2} r \mathcal{F}_{\phi(G)}$. Set

$$m = m^* + m_r \le \Big(\frac{r}{2} + 1\Big)\mathcal{F}_{\phi(G)}. \tag{5.6}$$

Taking z to be Za and Zb respectively where

$$Z = 4d\Big(\frac{r}{2} + 1\Big)\mathcal{F}_{\phi(G)} \ge 4dm, \tag{5.7}$$

we see there exist $a^* \in (Za + m^*\Omega) \cap \mathcal{L}(\phi(G))$ and $b^* \in (Zb + m^*\Omega) \cap \mathcal{L}(\phi(G))$.

By definition of successive minima there are $f_1, \cdots, f_r \in m_r\Omega$ that form a basis of $\mathcal{L}(\phi(G))$. Hence the $r+1$ vectors $a^*, a^* + f_1, \cdots, a^* + f_r \in Za + m^*\Omega + m_r\Omega \subset Za + m\Omega$ span W. At least one of them, denoted by a', lies out of the hyperplane $W_{k,l}$.

Denote $b' = b^* \in Za + m^*\Omega \subset Za + m\Omega$. Write $a' = Za + m\epsilon_a$, $b' = Zb + m\epsilon_b$ where $h_0(\epsilon_a), h_0(\epsilon_b) \leq 1$. Recall a', b' are both in the lattice $\mathcal{L}(\phi(G))$.

First of all, remark a' and b' are linearly independent. Actually $|(\epsilon_a)_j| \leq h_0(\epsilon_a) = 1, \forall j$, so

$$\begin{cases} a'_i = \dfrac{Z}{d_i} - m|(\epsilon_a)_i| \geq \dfrac{Z}{2} - m \geq \dfrac{2d-1}{4d} Z; \\ |a'_j| \leq \dfrac{Z}{d-d_i} + m|(\epsilon_a)_j| \leq Z + m \leq \dfrac{4d+1}{4d} Z, \forall j \neq i. \end{cases} \tag{5.8}$$

But similarly

$$\begin{cases} |b'_i| = m|(\epsilon_b)_i| \leq m \leq \dfrac{1}{4d} Z; \\ |b'_k| \geq \dfrac{Z}{d_k} - m|(\epsilon_b)_k| \geq \dfrac{Z}{2} - m \geq \dfrac{2d-1}{4d} Z. \end{cases} \tag{5.9}$$

Hence $\frac{|a'_i|}{|a'_k|} \geq \frac{2d-1}{4d+1} > \frac{1}{2d-1} \geq \frac{|b'_i|}{|b'_k|}$ when $d \geq 3$, which implies linear independence.

Second, suppose the 2-dimensional subspace $\mathbb{R}a' \oplus \mathbb{R}b'$ has a non-empty intersection with $W_{k,l} = \{w | Q_0(w, b) = 0\}$, then we claim that for all w in this intersection, $|w_i| \gtrsim h_0(w)$. Suppose $w = W_a a' + W_b b' = W_a Za + W_a m\epsilon_a + W_b Zb + W_b m\epsilon_b$. Then since $Q_0(a, b) = 0$ and by (5.4), $|Q_0(\epsilon_a, b)|, |Q_0(\epsilon_a, a)|, |Q_0(\epsilon_b, b)|$ and $|Q_0(\epsilon_b, b)|$ are all bounded by 1, so

$$\begin{aligned} 0 =& Q_0(w, b) = Q_0(W_a Za + W_a m\epsilon_a + W_b Zb + W_b m\epsilon_b, b) \\ \geq& |W_b Z Q_0(b, b)| - |W_a m Q_0(\epsilon_a, b)| - |W_b m Q_0(\epsilon_b, b)| \\ \geq& |W_b| Z Q_b - |W_a| m - |W_b| m, \end{aligned}$$

thus

$$\frac{|W_b|}{|W_a|} \leq \frac{m}{ZQ_b - m} \leq \frac{m}{\frac{1}{2}Z - m} \leq \frac{m}{2dm - m} = \frac{1}{2d-1}. \tag{5.10}$$

So

$$\begin{aligned} |Q_0(w, a)| =& |Q_0(W_a Za + W_a m\epsilon_a + W_b Zb + W_b m\epsilon_b, a)| \\ \geq& |W_a Z Q_0(a, a)| - |W_a m Q_0(\epsilon_a, a)| - |W_b m Q_0(\epsilon_b, a)| \\ \geq& |W_a| Z Q_a - |W_a| m - |W_b| m \\ \geq& \frac{1}{4} |W_a| Z - |W_a| \cdot \frac{Z}{4d} - \frac{|W_a|}{2d-1} \cdot \frac{Z}{4d} \\ =& \frac{2d-3}{8d-4} |W_a| Z. \end{aligned} \tag{5.11}$$

But on the other hand

$$
\begin{aligned}
2Q_0(w,a) =& d_i \cdot \frac{1}{d_i} w_i + \sum_{1\le j\le r_1+r_2, j\neq i} d_j w_j \left(-\frac{1}{d-d_i}\right) \\
=& \frac{d}{d-d_i} w_i - d_i w_i \cdot \frac{1}{d-d_i} - \sum_{1\le j\le r_1+r_2, j\neq i} d_j w_j \cdot \frac{1}{d-d_i} \\
=& \frac{d}{d-d_i} w_i - \frac{1}{d-d_i} \sum_{j=1}^{r_1+r_2} d_j w_j \\
=& \frac{d}{d-d_i} w_i.
\end{aligned}
\tag{5.12}
$$

Furthermore by (5.7) and (5.10),

$$
\begin{aligned}
h_0(w) \le& |W_a|Z + |W_b|Z + |W_a|m + |W_b|m \\
\le& \left(|W_a| + \frac{1}{2d-1}|W_a|\right)\left(Z + \frac{1}{4d}Z\right) = \frac{4d+1}{4d-2}|W_a|Z.
\end{aligned}
\tag{5.13}
$$

As $d \ge 3$ we eventually deduce from (5.11), (5.12) and (5.13) that

$$
\begin{aligned}
|w_i| =& 2 \cdot \frac{d-d_i}{d} |Q_0(w,a)| \\
\ge& 2 \cdot \frac{d-2}{d} \cdot \frac{2d-3}{8d-4} |W_a| Z \\
\ge& 2 \cdot \frac{d-2}{d} \cdot \frac{2d-3}{8d-4} \cdot \frac{4d-2}{4d+1} h_0(w) \\
=& \frac{(d-2)(2d-3)}{d(4d+1)} h_0(w) \\
\ge& \frac{1}{13} h_0(w).
\end{aligned}
\tag{5.14}
$$

As a', b' are in the lattice $\mathcal{L}(\phi(G))$, we can choose u and $\tilde{u}$ as following:
First let u be such that $\mathcal{L}(\phi(u)) = a'$, then

$$
\begin{cases}
h^{\mathrm{Mah}}(\phi(u)) & = h_0(a') \le Z + m \le (4d+1)(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}; \\
\log|\zeta_u^i| & = a'_i \ge \frac{2d-1}{4d} Z \ge (2d-1)(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}; \\
\log|\zeta_u^j| & = a'_j \le Z(-\frac{1}{d-d_i}) + m \\
& \le (-\frac{4d}{d-d_i}+1)m \\
& \le -3m < 0, \forall j \neq i.
\end{cases}
\tag{5.15}
$$

If $V_i \cong \mathbb{C}$, then $\zeta_{u^n}^i \notin \mathbb{R}$ as $\mathcal{L}(\phi(u^n)) = na' \notin W_{k,l}$ because of the way a' was chosen.
Case 1. If $(\mathbb{Z}a' \oplus \mathbb{Z}b') \cap W_{k,l} = \{0\}$. Choose $\tilde{u} \in G$ so that $\mathcal{L}(\phi(\tilde{u})) = a' + b'$. Then because a' and b' are linearly independent, u and $\tilde{u}$ are multiplicatively independent.

In this case, by remarks we made at the beginning of proof, either V_i is real or there is no non-trivial unit of the form $u^p\tilde{u}^q, (p,q) \neq (0,0)$ such that $\zeta^i_{u^p\tilde{u}^q} \in \mathbb{R}$ (because otherwise $pa' + q(a'+b') = p\mathcal{L}(\phi(u)) + q\mathcal{L}(\phi(\tilde{u})) \in W_{k,l}$, but by linear independence it doesn't vanish, thus contradictis the assumption $(\mathbb{Z}a' \oplus \mathbb{Z}b') \cap W_{k,l} = \{0\}$).

Notice $|b'_i| = |m(\epsilon_b))_i| \leq m \leq (\frac{r}{2}+1)\mathcal{F}_{\phi(G)}$. Then

$$\begin{cases} h^{\mathrm{Mah}}(\phi(\tilde{u})) & \leq (Z+m) + (Z+m) \leq (8d+2)(\dfrac{r}{2}+1)\mathcal{F}_{\phi(G)}, \\ \log|\zeta^i_{\tilde{u}}| & = a'_i + b'_i \geq a'_i - |b'_i| \geq (2d-2)(\dfrac{r}{2}+1)\mathcal{F}_{\phi(G)}. \end{cases} \tag{5.16}$$

Case 2. If $(\mathbb{Z}a' \oplus \mathbb{Z}b') \cap W_{k,l} \neq \{0\}$ then it is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$. Choose u so that $\mathcal{L}(\phi(u)) = a'$. But as $a' \notin W_{k,l}$, $(\mathbb{R}a' \oplus \mathbb{R}b') \cap W_{k,l}$ has dimension 1 and $(\mathbb{Z}a' \oplus \mathbb{Z}b') \cap W_{k,l}$ has to be cyclic. Let a'' be a generator, then by (5.14), $|a''_i| > \frac{1}{13}h^{\mathrm{Mah}}(a'') > 0$, without loss of generality assume $a''_i > 0$.

In this case, if there is $\zeta^i_{u^p\tilde{u}^q} \in \mathbb{R}$ for a non-trivial pair (p,q) then $p\mathcal{L}(\phi(u)) + q\mathcal{L}(\phi(\tilde{u})) \in (\mathbb{Z}a' \oplus \mathbb{Z}b') \cap W_{k,l}$, thus has to be a multiple of a''_i.

Case 2.i. If $h^{\mathrm{Mah}}(a'') \geq Z$ then we choose $\tilde{u}$ so that $\mathcal{L}(\phi(\tilde{u})) = a' + b'$. Then u and $\tilde{u}$ are always multiplicatively independent and the inequalities (5.16) still hold as in case 1. Moreover, if $\zeta^i_{u^p\tilde{u}^q} \in \mathbb{R}$ and $|\zeta^i_{u^p\tilde{u}^q}| > 1$, then $\log|\zeta^i_{u^p\tilde{u}^q}| = p\mathcal{L}(\phi(u)) + q\mathcal{L}(\phi(\tilde{u})) = na''_i$ for some $n \in \mathbb{N}$, thus

$$\log|\zeta^i_{u^p\tilde{u}^q}| \geq a''_i \geq \frac{1}{13}h^{\mathrm{Mah}}(a'') \geq \frac{1}{13}Z \geq \frac{d}{4}(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}. \tag{5.17}$$

Case 2.ii. If $h^{\mathrm{Mah}}(a'') < Z$ then there is $N \in \mathbb{N}$ such that $Nh^{\mathrm{Mah}}(a'') \in (Z, 2Z]$. There exists $\tilde{u} \in G$ such that $\mathcal{L}(\phi(\tilde{u})) = Na''$. Then $u, \tilde{u}$ are multiplicatively independent because $a' \notin W_{k,l}$ and $Na'' \in W_{k,l}$ are linearly independent.

In this case, if $\zeta^i_{u^p\tilde{u}^q}$ is real with $|\zeta^i_{u^p\tilde{u}^q}| \geq 1$ then $\log|\zeta^i_{u^p\tilde{u}^q}| = pa' + qNa'' \in W_{k,l}$. Because $a' \notin W_{k,l}$ and $a'' \in W_{k,l}$, $p = 0$ and $q > 0$. Thus

$$\log|\zeta^i_{u^p\tilde{u}^q}| \geq Na''_i \geq \frac{1}{13}Nh^{\mathrm{Mah}}(a'') \geq \frac{1}{13}Z \geq \frac{d}{4}(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}. \tag{5.17'}$$

Further,

$$\begin{cases} h^{\mathrm{Mah}}(\phi(\tilde{u})) & = Nh^{\mathrm{Mah}}(a'') \leq 2Z \leq 8d(\dfrac{r}{2}+1)\mathcal{F}_{\phi(G)}, \\ \log|\zeta^i_{\tilde{u}}| & = Na''_i \geq \dfrac{d}{4}(\dfrac{r}{2}+1)\mathcal{F}_{\phi(G)}. \end{cases} \tag{5.16'}$$

The proposition is established by combining (5.15), (5.16), (5.16'), (5.17) and (5.17'). $\square$

5.2 Approximation of an arithmetic progression

We will construct a sequence inside $\{\zeta^i_g | g \in G\}$ which resembles an arithmetic progression. To achieve this the following important result from Diophantine geometry is needed:

Theorem 5.3. (Baker-Wüstholz[1]) *Suppose $\alpha_k \in \mathbb{C}, k = 1, \cdots, N$ are non-zero algebraic numbers belonging to the same degree d number field and $\vartheta_k \in \mathbb{C}$ is a fixed natural logarithm of α_k for all k. Let*

$$h'_k = \max(h^{\mathrm{Mah}}(\alpha_k), |\log\vartheta_k|, 1).$$

Suppose a non-zero integer vector $(p_1, p_2, \cdots, p_N) \in \mathbb{Z}^N$ *satisfies* $\sum_{k=1}^N p_k\vartheta_k \neq 0$*, then*

$$-\log|\sum_{k=1}^N p_k\vartheta_k| \lesssim_{N,d} \big(\prod_{k=1}^N h_k'\big) \max(\log|p_1|, \log|p_2|, \cdots, \log|p_N|, \frac{1}{d}),$$

where the implied constant is explicitly determined by N *and* d*.*

Observe that, if $\mathrm{Im}\vartheta_k \in [0, 2\pi]$, then as $\log\vartheta_k = \log|\alpha_k| + \mathrm{Im}\vartheta_k$, $h_k' \lesssim \max(h^{\mathrm{Mah}}(\alpha_k), \log|\alpha_k|, 1)$. However $\log|\alpha_k| \leq \sum_{i=1}^d \log_+|\sigma_i(\alpha_k)| = h^{\mathrm{Mah}}(\alpha_k)$. Hence (a special case of) Baker-Wüstholz Theorem can be reformulated as:

Lemma 5.4. *Suppose* $\alpha_k \in \mathbb{C}, k = 1, \cdots, N$ *are non-zero algebraic numbers belonging to the same degree* d *number field and* $\vartheta_k \in \mathbb{C}$ *is a fixed natural logarithm of* α_k *with* $\mathrm{Im}\vartheta_k \in [0, 2\pi]$ *for all* $k = 1, \cdots, N$*. For any non-zero integer vector* $(p_1, p_2, \cdots, p_N) \in \mathbb{Z}^N$ *such that* $\sum_{k=1}^N p_k\vartheta_k \neq 0$*,*

$$-\log|\sum_{k=1}^N p_k\vartheta_k| \lesssim_{N,d} \big(\prod_{k=1}^N \max(h^{\mathrm{Mah}}(\alpha_k), 1)\big) \log\max(|p_1|, |p_2|, \cdots, |p_N|, 2),$$

where the implied constant is an explicit function in N *and* d*.*

Proposition 5.5. *For some effective constants* $\kappa_5(d)$ *and* $\kappa_6(d)$*, for all integers* $s \geq \kappa_5\mathcal{F}_{\phi(G)}$*, there exist* $a_0, a_1, \cdots, a_{s-1} \in G$ *and a number* Δ *(which is either real or complex depending on whether* $V_i \cong \mathbb{R}$ *or* $\mathbb{C}$*) such that:*

(1). $s^{-\kappa_6\mathcal{F}^2_{\phi(G)}} \leq |\Delta| \leq s^{-3}$*;*

(2). $h^{\mathrm{Mah}}(\phi(a_t)) \leq s^{10}, \forall t$*;*

(3). $|\zeta^i_{a_t} - (1 + t\Delta)| \leq s^{-1}|\Delta|, \forall t$*.*

Proof. Let u, $\tilde{u}$ be defined as in Proposition 5.2. Denote $\zeta^i_u = e^{\theta+2\pi\beta\mathrm{i}}, \zeta^i_{\tilde{u}} = e^{\tilde{\theta}+2\pi\tilde{\beta}\mathrm{i}}$, where $\theta, \tilde{\theta} > 0$ and $\beta, \tilde{\beta} \in [0, 1)$.

Let $\gamma = \theta/\tilde{\theta}$. Since $\frac{\tilde{\theta}}{\ln 2} = \log|\zeta^i_{\tilde{u}}| \geq \frac{1}{4}d(\frac{r}{2} + 1)\mathcal{F}_{\phi(G)}$ and $\frac{\theta}{\ln 2} = \log|\zeta^i_u| \leq h^{\mathrm{Mah}}(\phi(u)) \leq 9d(\frac{r}{2} + 1)\mathcal{F}_{\phi(G)}$, $\gamma \leq 36$.

Then $\forall n, m \in \mathbb{N}$, the i-th eigenvalue of $u^n\tilde{u}^{-m}$ is given by

$$\begin{aligned}\zeta^i_{u^n\tilde{u}^{-m}} =& e^{(\theta n-\tilde{\theta}m)+2\pi(\beta n-\tilde{\beta}m)\mathrm{i}}\\ =& e^{(\gamma\tilde{\theta}n-\tilde{\theta}m)+2\pi(\beta n-\gamma\tilde{\beta}n+\tilde{\beta}(\gamma n-m))\mathrm{i}}\\ =& e^{(\tilde{\theta}+2\pi\tilde{\beta}\mathrm{i})(\gamma n-m)+2\pi(\beta-\gamma\tilde{\beta})n\mathrm{i}}.\end{aligned} \tag{5.20}$$

Let $\Delta_0 = (\gamma, \beta - \gamma\tilde{\beta}) \in \mathbb{R}^2$. We slightly abuse notation and denote the projection of Δ_0 in $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ by Δ_0 as well. Then $\forall n \in \mathbb{N}$, $n\Delta_0 \in \mathbb{T}^2$ is represented by $(\gamma n, (\beta - \gamma\tilde{\beta})n)$. For all $P \geq 10$, $\mathbb{T}^2$ can be covered by at most P squares of side length $\frac{1}{\lfloor\sqrt{P}\rfloor} \leq \frac{2}{\sqrt{P}}$. By pigeonhole principle there exist two different $n_1, n_2 \in \{0, 1, \cdots, P\}$ such that $n_1\Delta_0$ and $n_2\Delta_0$ are in the same square. Let $n = |n_1 - n_2|$, then $1 \leq n \leq P$ and $n\Delta_0$ is in the neighborhood $\pi\big([-\frac{2}{\sqrt{P}}, \frac{2}{\sqrt{P}}]^2\big)$ of $(0, 0) \in \mathbb{T}^2$.

Now we are going to effectively bound the distance between $n\Delta_0$ and the origin using Baker-Wüsholtz Theorem (Lemma 5.4). We know for some $(m_1, m_2) \in \mathbb{Z}^2$,

$$\omega_1 := |\gamma n - m_1| \leq \frac{2}{\sqrt{P}}, \omega_2 := |(\beta - \gamma\tilde{\beta})n - m_2| \leq \frac{2}{\sqrt{P}}. \tag{5.21}$$

As $\frac{2}{\sqrt{P}} \leq 1$,

$$|m_1| \leq |\gamma n| + 1 \leq \gamma P + 1; \ |m_2| \leq |(\beta - \gamma\tilde{\beta})n| + 1 \leq |\beta - \gamma\tilde{\beta}|P + 1. \tag{5.22}$$

Notice

$$|\beta n - \tilde{\beta} m_1 - m_2| = |\tilde{\beta}(\gamma n - m_1) + (\beta - \tilde{\beta}\gamma)n - m_2| \leq \tilde{\beta}\omega_1 + \omega_2.$$

Multiply both sides by 2π, we get

$$|n \cdot 2\pi\beta\mathrm{i} - m_1 \cdot 2\pi\tilde{\beta}\mathrm{i} - m_2 \cdot 2\pi i| \leq 2\pi\tilde{(}\beta\omega_1 + \omega_2).$$

On the other hand,

$$|n\theta - m_1\tilde{\theta}| = |\tilde{\theta}(\gamma n - m_1)| \leq \tilde{\theta}\omega_1.$$

Add these two inequalities together:

$$|n(\theta + 2\pi\beta\mathrm{i}) - m_1(\tilde{\theta} + 2\pi\tilde{\beta}\mathrm{i}) - m_2 \cdot 2\pi\mathrm{i}| \leq (\tilde{\theta} + 2\pi\tilde{\beta})\omega_1 + 2\pi\omega_2. \tag{5.23}$$

But $\theta + 2\pi\beta\mathrm{i}$, $\tilde{\theta} + 2\pi\tilde{\beta}\mathrm{i}$ and $2\pi\mathrm{i}$ are respectively natural logarithms of ζ_u^i,$\zeta_{\tilde{u}}^i$ and 1.

Moreover we claim $n(\theta + 2\pi\beta\mathrm{i}) - m_1(\tilde{\theta} + 2\pi\tilde{\beta}\mathrm{i}) - m_2 \cdot 2\pi\mathrm{i} \neq 0$. Actually if this is not the case then $\zeta^i_{u^n\tilde{u}^{m_1}} = (\zeta_u^i)^n(\zeta_{\tilde{u}}^i)^{m_1} = 1$, but this would imply all of eigenvalues of $u^n\tilde{u}^{m_1}$ are equal to 1 since they are conjugate to each other. Thus $u^n\tilde{u}^{m_1}$ is identity, so $n = m_1 = 0$ by the multiplicative independence; but n is supposed to be positive.

Apply Lemma 5.4 to (5.23):

$$\begin{aligned}
& -\log\big((\tilde{\theta} + 2\pi\tilde{\beta})\omega_1 + 2\pi\omega_2\big) \\
\leq \quad & -\log|n(\theta + 2\pi\beta\mathrm{i}) - m_1(\tilde{\theta} + 2\pi\tilde{\beta}\mathrm{i}) - m_2 \cdot 2\pi\mathrm{i}| \\
\lesssim_d \quad & \max(h^{\mathrm{Mah}}(\zeta_u^i), 1)\max(h^{\mathrm{Mah}}(\zeta_{\tilde{u}}^i), 1)\log\max(|n|, |m_1|, |m_2|, 2) \\
= \quad & \max(h^{\mathrm{Mah}}(\phi(u)), 1)\max(h^{\mathrm{Mah}}(\phi(\tilde{u})), 1) \\
& \qquad\qquad \cdot \log\max(|n|, |m_1|, |m_2|, 2) \\
\lesssim_d \quad & \mathcal{F}^2_{\phi(G)}\log\max(P, \gamma P + 1, |\beta - \gamma\tilde{\beta}|P + 1, 2).
\end{aligned} \tag{5.24}$$

We used in the last inequality the fact that $h^{\mathrm{Mah}}(\phi(u)), h^{\mathrm{Mah}}(\phi(\tilde{u})) \lesssim_d \mathcal{F}_{\phi(G)}$ from Proposition 5.2 (using $r \leq d - 1$) and $\mathcal{F}_{\phi(G)} \geq \kappa_1(d)$ (by Remark 2.4).

When $P \geq 1$, $\max(P, \gamma P + 1, |\beta - \gamma\tilde{\beta}|P + 1, 2) \leq 40P$ as $\gamma \leq 36$ and $\beta, \tilde{\beta} < 1$, thus (5.24) implies there is an effective constant $\iota_1 = \iota_1(d)$ such that

$$(\tilde{\theta} + 2\pi\tilde{\beta})\omega_1 + 2\pi\omega_2 \geq P^{-\iota_1\mathcal{F}^2_{\phi(G)}}. \tag{5.25}$$

Define a sequence

$$a_t = (u^n\tilde{u}^{-m_1})^t \in G, \ t = 0, 1, \cdots. \tag{5.26}$$

By (5.20), $\zeta^i_{a_t} = e^{t\Delta}$ with $\Delta = (\tilde{\theta} + 2\pi\tilde{\beta}\mathrm{i})(\gamma n - m_1) + 2\pi((\beta - \gamma\tilde{\beta})n - m_2)\mathrm{i}$. Equivalently

$$\begin{cases} \mathrm{Re}\Delta = \tilde{\theta}(\gamma n - m_1) \\ \mathrm{Im}\Delta = 2\pi\tilde{\beta}(\gamma n - m_1) + 2\pi((\beta - \gamma\tilde{\beta})n - m_2). \end{cases} \tag{5.27}$$

We have two possibilities: either $(\tilde{\theta}+2\pi\tilde{\beta})\omega_1 \geq \frac{1}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2)$. Then $|\mathrm{Re}\Delta| = \tilde{\theta}\omega_1 \geq \frac{\tilde{\theta}}{\tilde{\theta}+2\pi\tilde{\beta}} \cdot \frac{1}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2)$. Since $\tilde{\beta} \in [0,1)$, $\tilde{\theta} = \log|\zeta^i_{\tilde{u}}| \gtrsim_d \mathcal{F}_{\phi(G)} \gtrsim_d 1$, we conclude by (5.25) in this case that

$$|\mathrm{Re}\Delta| \gtrsim_d P^{-\iota_1 \mathcal{F}^2_{\phi(G)}}.$$

Or $2\pi\omega_2 > \frac{2}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2)$, then $2\pi\tilde{\beta}\omega_1 \leq \frac{2\pi\tilde{\beta}}{\tilde{\theta}+2\pi\tilde{\beta}} \cdot \frac{1}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2) \leq \frac{1}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2)$. And

$$|\mathrm{Im}\Delta| \geq 2\pi\omega_2 - 2\pi\tilde{\beta}\omega_1 \geq \frac{1}{3}((\tilde{\theta}+2\pi\tilde{\beta})\omega_1+2\pi\omega_2) \geq \frac{1}{3}P^{-\iota_1 \mathcal{F}^2_{\phi(G)}}.$$

So in any case, there is an effective constant $\iota_2(d)$ such that

$$P^{-\iota_1 \mathcal{F}^2_{\phi(G)}} \leq \iota_2|\Delta|. \tag{5.28}$$

On the other hand, because $\omega_1, \omega_2 \leq \frac{2}{\sqrt{P}}$, $\tilde{\beta} < 1$, and

$$\tilde{\theta} \leq h^{\mathrm{Mah}}(\phi(\tilde{u})) \lesssim_d \mathcal{F}_{\phi(G)},$$

by (5.27) there is some $\iota_3(d)$ such that

$$|\Delta| \leq \iota_3 \mathcal{F}_{\phi(G)} P^{-\frac{1}{2}}. \tag{5.29}$$

Let $P = s^8$ where s is an integer such that $s \geq \kappa_5 \mathcal{F}_{\phi(G)}$ where $\kappa_5 = \kappa_5(d) \geq \max(\iota_3, \frac{\iota_2}{\kappa_1}, 1)$.
Now we verify all of the three claims from proposition.
(1). By (5.28) there exists an effective constant $\kappa_6 = \kappa_6(d)$ such that

$$\begin{aligned}|\Delta| \geq & {\iota_2}^{-1} P^{-\iota_1 \mathcal{F}^2_{\phi(G)}} \geq \frac{1}{\kappa_1\kappa_5} P^{-\iota_1 \mathcal{F}^2_{\phi(G)}} \geq s^{-1} s^{-8\iota_1 \mathcal{F}^2_{\phi(G)}} \\ \geq & s^{-\kappa_6 \mathcal{F}^2_{\phi(G)}},\end{aligned}$$

where we used $\mathcal{F}_{\phi(G)} \geq \kappa_1$.

On the other hand by as $s \geq \iota_3 \mathcal{F}_{\phi(G)}$. (5.29) implies $|\Delta| \leq sP^{-\frac{1}{2}} = s^{-3}$.
(2). We take only the first s terms in the sequence $\{a_t\}$. For all $0 \leq t \leq s-1$,

$$\begin{aligned}h^{\mathrm{Mah}}(\phi(a_t)) = & \ h^{\mathrm{Mah}}(\phi((u^n \tilde{u}^{-m_1})^t)) \leq t\big(|n| h^{\mathrm{Mah}}(\phi(u)) + |m_1| h^{\mathrm{Mah}}(\phi(\tilde{u}))\big) \\ \lesssim_d & \ s(|n|+|m_1|)\mathcal{F}_{\phi(G)}.\end{aligned}$$

As $1 \leq n \leq P$, by (5.22) $h^{\mathrm{Mah}}(\phi(a_t)) \lesssim_d sP\mathcal{F}_{\phi(G)} = s^9 \mathcal{F}_{\phi(G)}$. When κ_5 is made sufficiently large (but still effective), $h^{\mathrm{Mah}}(\phi(a_t)) \leq s^{10}$.
(3). Finally, if $|\Delta| \leq s^{-3}$ and $t < s$ then $|t\Delta| < 1$. So

$$|\zeta^i_{a_t} - (1+t\Delta)| = |e^{t\Delta} - (1+t\Delta)| = \Big|\sum_{k=2}^{\infty} \frac{(t\Delta)^k}{k!}\Big| \leq (\sum_{k=2}^{\infty}\frac{1}{k!})|t\Delta|^2 \leq |t\Delta|^2,$$

which is bounded by $s^2|\Delta|^2 = (s^2|\Delta|)|\Delta| \leq s^{-1}|\Delta|$. This completes the proof. □

5.3 Escape from a fixed line

In the last section we obtained a collection of eigenvalues of different elements from G in V_i which approximate the arithmetic progression $\{1+t\Delta\}_{t=1}^{s} \subset \mathbb{C}$. The direction of this progression is given by $\Delta \in C$.

When $V_i \cong \mathbb{C}$, it will turn out in a later part of this paper that in certain situations it may appear that some directions in $\mathbb{C}$ are "bad" in the sense that while embeded in $X = (\mathbb{R}^{r_1} \times \mathbb{C}^{r_2})/\Gamma$, they don't have the desired equidistribution property. So we will have to find a way to move our arithmetic progressions away from these exceptional directions, which is going to be the topic of the following discussion.

Proposition 5.6. *There exist effective constants $\kappa_7(d)$ and $\kappa_8(d)$ such that $\forall l \geq 2$, there are $b_1, b_2, \cdots, b_l \in G$ satisfying*

(1). $\forall k = 1, \cdots, l$, $h^{\mathrm{Mah}}(\phi(b_k)) \leq \kappa_7 \mathcal{F}_{\phi(G)}^2 l$;

(2). $1 \leq |\zeta_{b_k}^i| \leq l^{\kappa_8 \mathcal{F}_{\phi(G)}^2}$, $\forall k = 1, \cdots, l$;

(3). For any non-zero $\mathbb{R}$-linear form f on V_i, the number of the b_k's such that $f(\zeta_{b_k}^i) \leq \|f\|$ is bounded by 100.

We need a couple of lemmas to prove this result.

Lemma 5.7. *Assume $V \cong \mathbb{C}$ and f is a non-zero real form on V. For $\lambda > 0$, if complex numbers v, w satisfy $|v|, |w| \geq c > 0$ and $|f(v)|, |f(w)| \leq \lambda \|f\|$ then $\arg v$ and $\arg w$ are $\frac{\pi\lambda}{c}$-close up to a multiple of π, i.e. $|\arg v - \arg w - m\pi| < \frac{\pi\lambda}{c}$ for some $m \in \mathbb{Z}$.*

Proof. Without loss of generality, it suffices to prove in the case where $\|f\| = 1$.

Suppose $f(v) = f_1 \mathrm{Re} v + f_2 \mathrm{Im} v$, then $f_1^2 + f_2^2 = 1$. Let $z = f_2 + f_1 \mathrm{i}$. Observe

$$f(v) = \mathrm{Im}(zv) = |zv| |\sin(\arg z + \arg v)| = |v| |\sin(\arg z + \arg v)|.$$

So $|\sin(\arg z + \arg v)| \leq \frac{\lambda}{|v|} \leq \frac{\lambda}{c}$. Take $m_v \in \mathbb{Z}$ such that $\arg z + \arg v - m_v\pi \in [-\frac{\pi}{2}, \frac{\pi}{2})$ then $|\arg z + \arg v - m_v\pi| \leq \frac{\pi}{2} |\sin(\arg z + \arg v - m_v\pi)| = \frac{\pi}{2} |\sin(\arg z + \arg v)| \leq \frac{\pi\lambda}{2c}$. For the same reason $|\arg z + \arg w - m_w\pi| \leq \frac{\pi\lambda}{2c}$ for some integer m_w. Take the difference, we get $|\arg v - \arg w - (m_v - m_w)\pi| \leq \frac{\pi}{c}$. ☐

Lemma 5.8. *Suppose $F \subset \mathbb{C}$ is a number field of degree d which is already embedded into $\mathbb{C}$ and $\zeta, \rho \in F$ then there is a number field $\tilde{F} \subset \mathbb{C}$ of degree no more than $4d^2$ such that $\zeta/|\zeta|, \rho/|\rho| \in \tilde{F}$.*

Proof. $\overline{\zeta}$ and $\overline{\rho}$ belong to $\overline{F} \subset \mathbb{C}$, the complex conjugate of F which has degree d as well. The number field F' generated by F and $\overline{F}$ together is of degree no more than d^2. Since $|\zeta|^2 = \zeta\overline{\zeta} \in F'$ we see $|\zeta|$ is a quadratic element over F', and so is $|\rho|$. Hence the extension $\tilde{F} := F'(|\zeta|, |\rho|)$ of F is of degree at most 4. So $D := |\tilde{F} : \mathbb{Q}| \leq 4d^2$. It is clear that $\zeta/|\zeta|, \rho/|\rho| \in \tilde{F}$. ☐

Proof of Proposition 5.6. **Case 1** (trivial case). When $V_i \cong \mathbb{R}$, it suffices to take a single element $u \in G$, which was constructed in Proposition 5.2. Let $b_k = u, \forall 1 \leq k \leq l$. Then (1) is easily verified for sufficiently large $\kappa_7(d)$ since $h^{\mathrm{Mah}}(\phi(b_k)) = h^{\mathrm{Mah}}(\phi(u)) \lesssim_d \mathcal{F}_{\phi(G)}$ and $\mathcal{F}_{\phi(G)} \gtrsim_d 1$. We know $1 < |\zeta_u^i| \leq 2^{9d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}$, so (2) is satisfied if $\kappa_8 = \kappa_8(d)$ is large enough. Last, since any real linear form f is actually a scalar multiplication, $|f(\zeta_{b_k}^i)| = |f(\zeta_u^i)| = \|f\| \cdot |\zeta_u^i| > \|f\|$ for all k, part (3) follows.

Case 2 (main purpose). Suppose from now on $V_i \cong \mathbb{C}$. Take u and $\tilde{u}$ from Proposition 5.2. As in the proof of Proposition 5.5, set $\zeta_u^i = e^{\theta+2\pi\beta\mathrm{i}}, \zeta_{\tilde{u}}^i = e^{\tilde{\theta}+2\pi\tilde{\beta}\mathrm{i}}$, where $\theta, \tilde{\theta} > 0$ and $\beta, \tilde{\beta} \in [0, 1)$. Let $\theta = \gamma\tilde{\theta}$ then $\gamma \leq 36$ as before.

Construct a sequence of group elements

$$b_k = u^{k+J}\tilde{u}^{-\lceil\gamma k\rceil},\ k = 1, \cdots, l, \tag{5.30}$$

where

$$J := \lceil\theta^{-1}\iota_1\mathcal{F}^2_{\phi(G)} \ln l\rceil, \tag{5.31}$$

$\iota_1(d)$ being a sufficiently large effective constant to be decided later. Moreover we are going to suppose

$$l^{\frac{1}{2}\iota_1\mathcal{F}^2_{\phi(G)}} \geq e^{\tilde{\theta}}; \tag{5.32}$$

if ι_1 is large enough and $l \geq 2$, then this is always valid because $\tilde{\theta} \lesssim_d \mathcal{F}_{\phi(G)}$.

Since u and $\tilde{u}$ are multiplicatively independent, the b_k's are distinct. The action of b_k on V_i is characterized by

$$\zeta^i_{b_k} = e^{(k\theta+J\theta-\lceil\gamma k\rceil\tilde{\theta})+2\pi(k\beta+J\beta-\lceil\gamma k\rceil\tilde{\beta})\mathrm{i}}. \tag{5.33}$$

Now we check Proposition 5.6 part by part.

(1). Observe $h^{\mathrm{Mah}}(\phi(b_k)) \leq (k+J)h^{\mathrm{Mah}}(\phi(u))+\lceil\gamma k\rceil h^{\mathrm{Mah}}(\phi(\tilde{u}))$. Moreover, $\theta = \ln|\zeta^i_u| \gtrsim_d \mathcal{F}_{\phi(G)}$ effectively, which implies $J \lesssim_d \mathcal{F}_{\phi(G)} \ln l$. Since $h^{\mathrm{Mah}}(\phi(u)), h^{\mathrm{Mah}}(\phi(\tilde{u})) \lesssim_d \mathcal{F}_{\phi(G)}$, $k \leq l$ and $\gamma \leq 36$, we conclude

$$h^{\mathrm{Mah}}(\phi(b_k)) \lesssim_d \mathcal{F}^2_{\phi(G)} l,$$

with an effective implied constant $\kappa_7 = \kappa_7(d)$ for all $1 \leq k \leq l$.

(2). Because $\lceil\gamma k\rceil\tilde{\theta} \in [\gamma k\tilde{\theta}, \gamma k\tilde{\theta} + \tilde{\theta}) = [k\theta, k\theta + \tilde{\theta})$ and

$$J\theta \in [\iota_1\mathcal{F}^2_{\phi(G)} \ln l, \iota_1\mathcal{F}^2_{\phi(G)} \ln l + \theta),$$

by (5.33) $\ln|\zeta^i_{b_k}|$ is

$$k\theta + J\theta - \lceil\gamma k\rceil\tilde{\theta} \in [\iota_1\mathcal{F}^2_{\phi(G)} \ln l - \tilde{\theta}, \iota_1\mathcal{F}^2_{\phi(G)} \ln l + \theta). \tag{5.34}$$

So

$$\log|\zeta^i_{b_k}| \lesssim_d \mathcal{F}^2_{\phi(G)} \ln l$$

and the second condition is satisfied by some effective κ_8.

(3). Suppose there are more than one b_k such that $f(\zeta^i_{b_k}) \leq \|f\|$ (otherwise we are done). Apply Lemma 5.7 to two such elements $\zeta^i_{b_k}$ and $\zeta^i_{b_{k'}}$, by (5.34) there is $m \in \mathbb{Z}$ such that $\big|2\pi(k\beta + J\beta - \lceil\gamma k\rceil\tilde{\beta}) - 2\pi(k'\beta + J\beta - \lceil\gamma k'\rceil\tilde{\beta}) - m\pi\big|$ is at most

$$\frac{\pi}{\min(|\zeta^i_{b_k}|, |\zeta^i_{b_{k'}}|)} \leq \pi e^{-(\iota_1\mathcal{F}^2_{\phi(G)} \ln l-\tilde{\theta})} = \pi e^{\tilde{\theta}} l^{-\iota_1\mathcal{F}^2_{\phi(G)}}.$$

If we denote $n_1 = k - k'$ and $n_2 = -\lceil\gamma k\rceil + \lceil\gamma k'\rceil$, this rewrites

$$|n_1 \cdot 2\pi\beta\mathrm{i} + n_2 \cdot 2\pi\tilde{\beta}\mathrm{i} - m \cdot \pi\mathrm{i}| \leq \pi e^{\tilde{\theta}} l^{-\iota_1\mathcal{F}^2_{\phi(G)}}. \tag{5.35}$$

Assume first $n_1 \cdot 2\pi\beta + n_2 \cdot 2\pi\tilde{\beta} - m\pi \neq 0$.

It is easy to see

$$|n_1| \leq l;\ |n_2| \leq \gamma l + 1 \leq 40l, \tag{5.36}$$

(as $\gamma \leq 36$). Hence by assumption (5.32),

$$\begin{aligned} |m| \leq & \pi^{-1}(2\pi|n_1| + 2\pi|n_2| + \pi e^{\tilde{\theta}} l^{-\iota_1 \mathcal{F}^2_{\phi(G)}}) \\ \leq & \pi^{-1}(2\pi|n_1| + 2\pi|n_2| + \pi) \leq 100l. \end{aligned} \tag{5.37}$$

Notice $2\pi\beta\mathrm{i}$, $2\pi\tilde{\beta}\mathrm{i}$ and πi are respectively natural logarithms of the numbers $\zeta^i_u/|\zeta^i_u|$, $\zeta^i_{\tilde{u}}/|\zeta^i_{\tilde{u}}|$ and -1.

By Lemma 5.8, $\zeta^i_u/|\zeta^i_u|$ and $\zeta^i_{\tilde{u}}/|\zeta^i_{\tilde{u}}|$ are algebraic numbers in a number field $\tilde{K} \subset \mathbb{C}$ of degree $D \leq 4d^2$.

Let $\tilde{h}^{\mathrm{Mah}}$ be the logarithmic Mahler measure on $\tilde{K}$. Then for $\lambda \in K$, the absolute logarithmic height $h(d) = \frac{\tilde{h}^{\mathrm{Mah}}(\lambda)}{D} = \frac{h^{\mathrm{Mah}}(\lambda)}{d}$. Hence $\tilde{h}^{\mathrm{Mah}}(\zeta^i_u) = \tilde{h}^{\mathrm{Mah}}(\overline{\zeta^i_u}) = \frac{D}{d} h^{\mathrm{Mah}}(\phi(u)) \leq \frac{4d^2}{d} \cdot 9d(\frac{r}{2} + 1)\mathcal{F}_{\phi(G)} \lesssim_d \mathcal{F}_{\phi(G)}$ and by (2.5) $\tilde{h}^{\mathrm{Mah}}(|\zeta^i_u|) = \frac{1}{2}\tilde{h}^{\mathrm{Mah}}(\zeta^i_u\overline{\zeta^i_u}) \leq \frac{1}{2}(\tilde{h}^{\mathrm{Mah}}(\zeta^i_u) + \tilde{h}^{\mathrm{Mah}}(\overline{\zeta^i_u})) \lesssim_d \mathcal{F}_{\phi(G)}$. So $\tilde{h}^{\mathrm{Mah}}(\zeta^i_u/|\zeta^i_u|) \lesssim_d \mathcal{F}_{\phi(G)}$, similarly $\tilde{h}^{\mathrm{Mah}}(\zeta^i_{\tilde{u}}/|\zeta^i_{\tilde{u}}|) \lesssim_d \mathcal{F}_{\phi(G)}$. All the implied constants are effective here.

So Lemma 5.4 applies, by (5.35),

$$\begin{aligned} & -\log(\pi e^{\tilde{\theta}} l^{-\iota_1 \mathcal{F}^2_{\phi(G)}}) \\ \leq \quad & -\log|n_1 \cdot 2\pi\beta\mathrm{i} + n_2 \cdot 2\pi\tilde{\beta}\mathrm{i} - m \cdot \pi\mathrm{i}| \\ \lesssim_D \quad & \max(\tilde{h}^{\mathrm{Mah}}(\frac{\zeta^i_u}{|\zeta^i_u|}), 1)\max(\tilde{h}^{\mathrm{Mah}}(\frac{\zeta^i_{\tilde{u}}}{|\zeta^i_{\tilde{u}}|}), 1) \\ & \qquad \cdot \log\max(|n_1|, |n_2|, |m|, 2) \\ \lesssim_d \quad & \mathcal{F}^2_{\phi(G)} \log(100l), \end{aligned} \tag{5.38}$$

with an effective implied constant. In the last step we used the fact there there are only finitely many choices for $D \leq 4d^2$ for a fixed d and estimates (5.36), (5.37).

By (5.38) and assumption (5.32), $l^{-\frac{1}{2}\iota_1 \mathcal{F}^2_{\phi(G)}} \geq e^{\tilde{\theta}} l^{-\iota_1 \mathcal{F}^2_{\phi(G)}} \geq \frac{1}{\pi} l^{-\iota_2 \mathcal{F}^2_{\phi(G)}}$, where both $\iota_2 = \iota_2(d)$ is an effective constant arising from (5.38). But if we pick $l \geq 2$ and ι_1 sufficently large with respect to ι_2 then the inequality cannot be true. Contradiction.

Therefore $n_1 \cdot 2\pi\beta + n_2 \cdot 2\pi\tilde{\beta} - m\pi$ has to vanish.

Since $n_1 \cdot 2\pi\beta + n_2 \cdot 2\pi\tilde{\beta}$ is one argument of $(\zeta^i_u)^{n_1}(\zeta^i_{\tilde{u}})^{n_2}$, so this restriction actually says $\zeta^i_{b_k b^{-1}_{k'}} = \zeta^i_{u^{n_1}\tilde{u}^{n_2}}$ is real.

Consider the set $\Lambda = \{(p, q) \in \mathbb{Z}^2 | (\zeta^i_u)^p(\zeta^i_{\tilde{u}})^q \in \mathbb{R}\}$, which is a subgroup of $\mathbb{Z}^2$. So Λ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}^2$ if it is not trivial. Assume $\Lambda \cong \mathbb{Z}^2$ then it is a lattice in $\mathbb{Z}^2$ and there exists a number n such that $(n, 0) \in \Lambda$, i.e. $\zeta^i_{u^n} = (\zeta^i_h)^n \in \mathbb{R}$, which contradicts Proposition 5.2. Therefore Λ has to be infinite cyclic. Fix a generator $(p, q) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$ of Λ, then (n_1, n_2) is a non-trivial multiple of (p, q). So $\log|\zeta^i_{u^{n_1}\tilde{u}^{n_2}}|$ is a non-trivial multiple of $\log|\zeta^i_{u^p\tilde{u}^q}|$, which is at least $\frac{1}{4}d(\frac{r}{2} + 1)\mathcal{F}_{\phi(G)}$ by Proposition 5.2.

But by (5.34), $\log|\zeta^i_{b_k b^{-1}_{k'}}| \leq \log|e^{\tilde{\theta}} e^{\theta}| = \log|\zeta^i_u| + \log|\zeta^i_{\tilde{u}}| \leq h^{\mathrm{Mah}}(\phi(u)) + h^{\mathrm{Mah}}(\phi(\tilde{u})) < 18d(\frac{r}{2} + 1)\mathcal{F}_{\phi(G)}$. Thus there are at most 72 choices of $(n_1, n_2) = (k - k', -\lceil\gamma k\rceil + -\lceil\gamma k'\rceil)$. Hence if we fix k, there are at most 72 other indices k' such that $f(\zeta^i_{b_{k'}}) \leq \|f\|$, this establishes the last part of proposition. $\square$

6 Measure-theoretical results

We will effectively construct some element in the convex hull of $\{g.\tau, g \in G\}$ which approximately dominates some positive multiple of the Lebesgue measure m on X.

6.1 Fourier coefficients on X

Assuming Conditions 2.8, 1.8, in this part we are going to study the behavior of the measure ν in Proposition 4.16 under the G action. We will show in an effective way that certain averages of measures in the orbit $G.\nu$ will approach a positive multiple of the Lebesgue measure. For this purpose we will need the L^2-bound from Proposition 4.16 as well as the phenomena described in Section 5.

Recall $\psi \in M_d(\mathbb{R})$ is a linear conjuagtion between $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $X = \mathbb{R}^d/\Gamma$ as described in Proposition 2.13.

Definition 6.1. *Given a measure γ on X, the Fourier coefficient of γ at frequency $\xi \in X^* = \psi_*(\mathbb{Z}^d)$ is*

$$\hat{\gamma}(\xi) = \int_{x\in X} e(\xi(x))\mathrm{d}\gamma(x)$$

where $e(\beta) = e^{-2\pi\beta\mathrm{i}}$.

When γ comes from a measure on $\mathbb{T}^d$, the following lemma is obvious.

Lemma 6.2. *For all measures γ on $\mathbb{T}^d$ and all $\mathbf{q} \in \mathbb{Z}^d$, $\widehat{\psi_*\gamma}(\psi_*\mathbf{q}) = \hat{\psi}(\mathbf{q})$.*

Notation 6.3. *Given a measure ν on X and an index $i \in I$, for $n, s, l \in \mathbb{N}$ such that $s \geq \kappa_5\mathcal{F}_{\phi(G)}$, $l \geq 2$ where κ_5 is as in Propositions 5.5, we write*

$$\nu^i_{n,s,l} = \frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l} \times_{\phi(u^n a_t b_k)}.\nu, \tag{6.1}$$

with $u, a_t, b_k \in G$ respectively defined in Propositions 5.2, 5.5 and 5.6 and $\times_{\phi(u^n a_t b_k)}.\nu$ denoting the pushforward of ν by $\times_{\phi(u^n a_t b_k)}$ on X.

With ν and i constructed in Proposition 4.16, we hope to control the sizes of the Fourier coefficients of $\nu^i_{n,s,l}$ when the parameters n, s, and l are carefully chosen. The first step to do this is the following estimate:

Proposition 6.4. *Assuming Conditions 2.8, 1.8, let ν and i be as in Proposition 4.16 and adopt other notations from that proposition as well. Suppose*

$$\mathcal{M}_\psi A|\zeta^i_u|^n|\Delta|l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^{-R} \leq \frac{1}{4}, \tag{6.2}$$

then $|\widehat{\nu^i_{n,s,l}}(\xi)|^2 \le \sum_{c=1}^5 L_c$ *where*

$$\begin{aligned}
L_1 =& 9 \cdot 2^{-d_i \delta T} &(6.3a)\\
L_2 =& 2\pi\sqrt{d}\mathcal{M}_\psi A 2^{s^{10}+\kappa_7 \mathcal{F}^2_{\phi(G)}} l^{-R} &(6.3b)\\
L_3 =& 2\pi\sqrt{d}\mathcal{M}_\psi A|\zeta_u^i|^n s^{-1}|\Delta| l^{\kappa_8 \mathcal{F}^2_{\phi(G)}} 2^{-R} &(6.3c)\\
L_4 =& \frac{d}{2} 2^{\frac{d(d-1)}{2}\kappa_3 \mathcal{F}_{\phi(G)}} \mathcal{M}_\psi^{d-1} A^{d-1} |\zeta_u^i|^{-n} s^{-1} |\Delta|^{-1} 2^{R+T} &(6.3d)\\
L_5 =& \frac{100}{l} &(6.3e)
\end{aligned}$$

Lemma 6.5. *For* $\beta \in (-\frac{1}{4}, \frac{1}{4})$ *and* $s \in \mathbb{N}$, $|\sum_{t=0}^{s-1} e(t\beta)| \le (2|\beta|)^{-1}$.

Proof. It is an easy fact that when $b \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $|\sin b| \ge \frac{2}{\pi}|b|$. Therefore $|\sum_{t=0}^{s-1} e(t\beta)| = |\frac{1-e(s\beta)}{1-e(\beta)}| \le \frac{2}{|1-\exp(2\pi\beta \mathrm{i})|} \le \frac{2}{|\sin 2\pi\beta|} \le \frac{2}{\frac{2}{\pi}\cdot 2\pi|\beta|} = (2|\beta|)^{-1}$. ☐

Proof of Proposition 6.4. Recall $\nu = \mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \nu_x$ where $\mathrm{m}_{\tau,B}$ is a probability measure on X and each ν_x is supported on $x + B$.

For any fixed positive number A, for all $\xi \in X^*\backslash\{0\}$ with $|\xi| \le A$,

$$\begin{aligned}
|\widehat{\nu^i_{n,s,l}}(\xi)|^2 =& \Big| \int_X e(\xi(y)) \cdot \frac{1}{sl} \mathrm{d}\big(\sum_{t=0}^{s-1}\sum_{k=1}^{l} \times_{\phi(u^n a_t b_k)} .\nu\big)(y) \Big|^2 \\
=& \Big| \int_X \frac{1}{sl} \sum_{t=0}^{s-1}\sum_{k=1}^{l} e\big(\xi(\times_{\phi(u^n a_t b_k)} .y)\big) \mathrm{d}\nu(y) \Big|^2 \\
=& \Big| \mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \frac{1}{sl} \sum_{t=0}^{s-1}\sum_{k=1}^{l} \int_{x+B} e\big(\xi(\times_{\phi(u^n a_t b_k)} .y)\big) \mathrm{d}\nu_x(y) \Big|^2
\end{aligned}$$

By Cauchy-Schwarz,

$$\begin{aligned}
|\widehat{\nu^i_{n,s,l}}(\xi)|^2 \le& \mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \frac{1}{sl} \sum_{t=0}^{s-1}\sum_{k=1}^{l} \Big| \int_{x+B} e\big(\xi(\times_{\phi(u^n a_t b_k)} .y)\big) \mathrm{d}\nu_x(y) \Big|^2 \\
=& \mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \frac{1}{sl} \sum_{t=0}^{s-1}\sum_{k=1}^{l} \Big(\int_{x+B} e\big(\xi(\times_{\phi(u^n a_t b_k)} .y)\big) \mathrm{d}\nu_x(y) \Big) \\
& \overline{\Big(\int_{x+B} e\big(\xi(\times_{\phi(u^n a_t b_k)} .y)\big) \mathrm{d}\nu_x(y) \Big)}
\end{aligned}$$

Using the simple fact that $\overline{e(\beta)} = e(-\beta), \forall \beta \in \mathbb{R}$, we obtain

$$\begin{aligned}
|\widehat{\nu^i_{n,s,l}}(\xi)|^2 =& \mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \frac{1}{sl} \sum_{t=0}^{s-1}\sum_{k=1}^{l} \\
& \iint_{y,z \in x+B} e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y - z)\big)\Big) \mathrm{d}\nu_x(y) d\nu_x(z).
\end{aligned} \tag{6.4}$$

Next, we decompose $x+B$ by $x+\mathcal{Q}_{T\mathbf{1}_i}B$:

$$\begin{aligned}
&|\widehat{\nu^i_{n,s,l}}(\xi)|^2\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B}\\
&\iint_{\substack{y\in x+P\\ z\in x+Q}}\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)}.(y-z)\big)\Big)\mathrm{d}\nu_x(y)d\nu_x(z)\\
=&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B\\ \bar{P}\cap\bar{Q}\neq\emptyset}}\mathcal{I}_{P,Q,x}+\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B\\ \bar{P}\cap\bar{Q}=\emptyset}}\mathcal{I}_{P,Q,x},
\end{aligned}\tag{6.5}$$

where

$$\mathcal{I}_{P,Q,x}=\iint_{\substack{y\in x+P\\ z\in x+Q}}\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)}.(y-z)\big)\Big)\mathrm{d}\nu_x(y)d\nu_x(z)\geq 0.\tag{6.6}$$

From the construction of the partition $\mathcal{Q}_{T\mathbf{1}_i}B$ we know for each atom P in it there are at most $9=3^2$ atoms Q such that the closures $\bar{P}$ and $\bar{Q}$ intersect (when $V_i\cong\mathbb{R}$, there are actually at most 3 such atoms). So each atom appear in at most 9 adjacent pairs (P,Q) as P (and as Q as well). Since

$$\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)}.(y-z)\big)\Big)\Big|\leq\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}1\Big|=1,$$

we have

$$\begin{aligned}
&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B\\ \bar{P}\cap\bar{Q}\neq\emptyset}}\mathcal{I}_{P,Q,x}\\
\leq&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B\\ \bar{P}\cap\bar{Q}\neq\emptyset}}\nu_x(x+P)\nu_x(x+Q)\\
\leq&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}\sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B\\ \bar{P}\cap\bar{Q}\neq\emptyset}}\nu_x^2(x+P)\qquad\text{(Cauchy-Schwarz)}\\
\leq&\mathbb{E}_{\mathrm{m}_{\tau,B}(x)}9\sum_{P\in\mathcal{Q}_{T\mathbf{1}_i}B}\nu_x^2(x+P)\\
\leq&9\cdot 2^{-d_i\delta T}=L_1,
\end{aligned}\tag{6.7}$$

where the last step follows from the L^2-bound in Propoperty 4.16.

From now on we suppose two points $y,z\in X$ and two atoms $P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B$ are such that

$$\bar{P}\cap\bar{Q}=\emptyset,\ y\in x+P,\ z\in x+Q.\tag{6.8}$$

In this case the orthogonal projection of the distance vector $y-z$ in subspace V_i, denoted by $(y-z)_i$, has size at least $2^{-R-T}\mathcal{S}_\Gamma$, which is the length of the sides of atoms from $\mathcal{Q}_{T\mathbf{1}_i}B$ in V_i-direction. On the other hand $|y-z|\leq\sqrt{d}2^{-R}\mathcal{S}_\Gamma$ because y, z belongs to the same d-dimensional cube $x+B$ whose sides have length $2^{-R}\mathcal{S}_\Gamma$.

Thus the difference $(y-z)' \overset{\text{def}}{=} (y-z)-(y-z)_i$, which is the orthogonal projection into $\oplus_{j\neq i}V_j$, has size at most $\sqrt{d}2^{-R}\mathcal{S}_\Gamma$. Recall $|\zeta_u^i| \geq 2^{d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}$ and $|\zeta_u^j| \leq 1, \forall j\neq i$; $\log|\zeta_{a_t}^j| \leq \log h^{\mathrm{Mah}}(\phi(a_t)) \leq s^{10}$ and similarly $|\zeta_{b_k}^j| \leq \kappa_7 \mathcal{F}_{\phi(G)}^2 l$. Thus

$$|\times_{\phi(u^n a_t b_k)} .(y-z)'| \leq 2^{s^{10}+\kappa_7\mathcal{F}_{\phi(G)}^2 l}\cdot\sqrt{d}2^{-R}.$$

$e:\mathbb{R}\mapsto\mathbb{S}^1\subset\mathbb{C}$ is Lipschitz continuous with 2π as Lipschitz constant. As $\|\xi\|\leq|\xi|\cdot\|\psi^{-1}\|\leq A\mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}$, $e(\xi(\cdot))$ has Lipschitz constant $2\pi A\mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}$ and

$$\begin{aligned}
&\Big|e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)\big)\Big)-e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)_i\big)\Big)\Big|\\
\leq&2\pi A\mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}\cdot|\times_{\phi(u^n a_t b_k)} .(y-z)'|\\
\leq&2\pi\sqrt{d}\mathcal{M}_\psi A 2^{s^{10}+\kappa_7\mathcal{F}_{\phi(G)}^2 l-R}=L_2.
\end{aligned}$$

Hence

$$\begin{aligned}
&\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)\big)\Big)\Big|\\
\leq&\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)_i\big)\Big)\Big|+L_2.
\end{aligned}\tag{6.9}$$

We now study $\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)_i\big)\Big)\Big|$. First of all observe since $(y-z)_i$ lies in V_i, which is regarded as $\mathbb{R}$ or $\mathbb{C}$, $\times_{\phi(u^n a_t b_k)}.(y-z)_i=\zeta_u^n\zeta_{a_t}^i\zeta_{b_k}^i(y-z)_i$.

By Proposition 5.5 $|\zeta_{a_t}^i-(1+t\Delta)|\leq s^{-1}|\Delta|$ with $s^{-\kappa_6}\mathcal{F}_{\phi(G)}^2\leq|\Delta|\leq s^{-3}$. And the other factor $|\zeta_{b_k}^i|\leq l^{\kappa_8\mathcal{F}_{\phi(G)}^2}$. Once again because of Lipschitz continuity,

$$\begin{aligned}
&\Big|e\Big(\xi\big(\times_{\phi(u^n a_t b_k)} .(y-z)_i\big)\Big)-e\Big(\xi\big((\zeta_u^i)^n(1+t\Delta)\zeta_{b_k}^i(y-z)_i\big)\Big)\Big|\\
\leq&2\pi A\mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}\cdot|\zeta_u^i|^n|\zeta_{a_t}^i-(1+t\Delta)||\zeta_{b_k}^i|\cdot|(y-z)_i|\\
\leq&2\pi\sqrt{d}\mathcal{M}_\psi A|\zeta_u^i|^n s^{-1}|\Delta|l^{\kappa_8\mathcal{F}_{\phi(G)}^2}2^{-R}=L_3.
\end{aligned}$$

So

$$(6.9)\leq\Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}e\Big(\xi\big((\zeta_u^i)^n(1+t\Delta)\zeta_{b_k}^i(y-z)_i\big)\Big)\Big|+L_2+L_3.\tag{6.10}$$

Define a $\mathbb{R}$-linear form $f(v):=\xi\big((\zeta_u^i)^n\Delta v(y-z)_i\big)$ on either $\mathbb{R}$ or $\mathbb{C}$ depending on whether $V_i\cong\mathbb{R}$ or $\mathbb{C}$. Then $\|f\|=\big\|\xi\big|_{V_i}\big\|\cdot|\zeta_u^i|^n|\Delta|\cdot|(y-z)_i|$. However, by Lemma 3.5, $\big\|\xi\big|_{V_i}\big\|\geq d^{-1}2^{-\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}_\psi^{-(d-1)}A^{-(d-1)}\mathcal{S}_\Gamma^{-1}$. Thus by Proposition 5.6, there is a subset $J\subset\{1,\cdots,l\}$ of cardinality at most 100 such that $\forall k\in\{1,\cdots,l\}\backslash J$,

$$\begin{aligned}
&|\xi\big((\zeta_u^i)^n\Delta\zeta_{b_k}^i(y-z)_i\big)|\\
=&|f(\zeta_{b_k}^i)|\geq\|f\|\\
\geq&\|\xi|_{V_i}\|\cdot|\zeta_u^i|^n|\Delta|\cdot|(y-z)_i|\mathcal{S}_\Gamma\\
\geq&d^{-1}2^{-\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}_\psi^{-(d-1)}A^{-(d-1)}|\zeta_u^i|^n|\Delta|2^{-R-T}\\
=&\frac{1}{2s}L_4^{-1}.
\end{aligned}\tag{6.11}$$

Here we used the fact $|(y-z)_i| \geq 2^{-R-T}\mathcal{S}_\Gamma$.

On the other hand, it follows from assumption (6.2) that

$$\begin{aligned} |\xi((\zeta_u^i)^n \Delta \zeta_{b_k}^i (y-z)_i))| \leq & A\mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} |(\zeta_u^i)^n \Delta \zeta_{b_k}^i (y-z)_i| \\ = & \mathcal{M}_\psi A |\zeta_u^i|^n |\Delta| l^{\kappa_8 \mathcal{F}^2_{\phi(G)}} 2^{-R} \leq \frac{1}{4}. \end{aligned} \tag{6.12}$$

Moreover notice that

$$\begin{aligned} & \Big|\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l} e\Big(\xi\big((\zeta_u^i)^n(1+t\Delta)\zeta_{b_k}^i(y-z)_i\big)\Big)\Big| \\ \leq & \frac{1}{l}\sum_{k=1}^{l}\Big|\frac{1}{s}\sum_{t=0}^{s-1} e\Big(\xi\big((\zeta_u^i)^n(1+t\Delta)\zeta_{b_k}^i(y-z)_i\big)\Big)\Big| \\ = & \frac{1}{l}\sum_{k=1}^{l}\Big|e\Big(\xi\big((\zeta_u^i)^n\zeta_{b_k}^i(y-z)_i\big)\Big)\cdot\frac{1}{s}\sum_{t=0}^{s-1} e\Big(\xi\big((\zeta_u^i)^n t\Delta\zeta_{b_k}^i(y-z)_i\big)\Big)\Big| \\ = & \frac{1}{l}\sum_{k=1}^{l}\Big|\frac{1}{s}\sum_{t=0}^{s-1} e\Big(t\xi\big((\zeta_u^i)^n \Delta\zeta_{b_k}^i(y-z)_i\big)\Big)\Big|, \end{aligned} \tag{6.13}$$

by separating the exception set J from generic indices we get

$$\begin{aligned} (6.13) \leq & \frac{1}{l}\Big(100 + \sum_{k\in\{1,\cdots,l\}\setminus J}\Big|\frac{1}{s}\sum_{t=0}^{s-1} e\Big(t\xi\big((\zeta_u^i)^n\Delta\zeta_{b_k}^i(y-z)_i\big)\Big)\Big|\Big) \\ \leq & \frac{1}{l}\Big(100 + \sum_{k\in\{1,\cdots,l\}\setminus J}\frac{1}{s}\Big(2\Big|\xi\big((\zeta_u^i)^n\Delta\zeta_{b_k}^i(y-z)_i)\big)\Big|\Big)^{-1}\Big), \end{aligned} \tag{6.14}$$

where Lemma 6.5 was applied in the last step in light of condition (6.12).

Making use of (6.11), we obtain from (6.14) that

$$(6.13) \leq \frac{1}{l}\big(100 + l\cdot\frac{1}{s}(2\cdot\frac{1}{2s}L_4^{-1})^{-1}\big) = \frac{100}{l} + L_4 = L_4 + L_5 \tag{6.15}$$

Plug (6.15) into (6.10), we get

$$(6.9) \leq L_2 + L_3 + L_4 + L_5. \tag{6.16}$$

under the assumptions (6.2) and (6.8).

Plug (6.16) into (6.6), it follows from the fact $|\nu_x| \leq 1$ almost everywhere (see Proposition 4.16) that $\mathcal{I}_{P,Q,x} \leq L_2 + L_3 + L_4 + L_5$ for a.e. x if $\bar{P}\cap\bar{Q} = 0$ thus

$$\mathbb{E}_{\mathrm{m}_{\tau,B}(x)} \sum_{\substack{P,Q\in\mathcal{Q}_{T\mathbf{1}_i}B \\ \bar{P}\cap\bar{Q}=\emptyset}} \mathcal{I}_{P,Q,x} \leq L_2 + L_3 + L_4 + L_5. \tag{6.17}$$

The proposition is verified by combining (6.5), (6.7) and (6.17). $\square$

6.2 Manipulation of parameters

We want to show that if the parameters δ, T and n, s, l are properly chosen then

$$|\widehat{\nu^i_{n,s,l}}(\xi)|^2 \lesssim_d A^{-1} \tag{6.18}$$

for any non-trivial frequency ξ with $|\xi| \leq A$ as long as A is a large number but still sufficiently small compared to R.

Actually by Proposition 6.4 this would follow from the following collection of inequalities:

$$\begin{cases} \mathcal{M}_\psi A|\zeta^i_u|^n|\Delta|l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^{-R} \leq \dfrac{1}{4} & (6.2) \\ 2^{-d_i\delta T} \lesssim A^{-1} & (6.19a) \\ \mathcal{M}_\psi A2^{s^{10}+\kappa_7\mathcal{F}^2_{\phi(G)}l-R} \lesssim A^{-1} & (6.19b) \\ \mathcal{M}_\psi A|\zeta^i_u|^n s^{-1}|\Delta|l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^{-R} \lesssim A^{-1} & (6.19c) \\ 2^{\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}^{d-1}_\psi A^{d-1}|\zeta^i_u|^{-n}s^{-1}|\Delta|^{-1}2^{R+T} \lesssim A^{-1} & (6.19d) \\ l^{-1} \lesssim A^{-1} & (6.19e) \end{cases}$$

First of all, set

$$l = A = \lceil 2^{d_i\delta T}\rceil \tag{6.20}$$

so that (6.19a) and (6.19e) are satisfied. Multiplying (6.2) by (6.19d) gives

$$2^{\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}^d_\psi A^d s^{-1} l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^T \lesssim A^{-1}.$$

So s should be such that

$$\begin{aligned} s \gtrsim & 2^{\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}^d_\psi A^{d+1} l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^T \\ \approx & 2^{\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}^d_\psi 2^{\left(1+(d+1+\kappa_8\mathcal{F}^2_{\phi(G)})d_i\delta\right)T}. \end{aligned}$$

We take the critical setting

$$s = \lceil 2^{\frac{d(d-1)}{2}\kappa_3\mathcal{F}_{\phi(G)}}\mathcal{M}^d_\psi 2^{(1+(d+1+\kappa_8\mathcal{F}^2_{\phi(G)})d_i\delta)T}\rceil. \tag{6.21}$$

Now the parameter Δ is determined by s together with the index i. Take n so that both sides of (6.2) are almost equal up to a multiplicative error:

$$\mathcal{M}_\psi A|\zeta^i_u|^n|\Delta|l^{\kappa_8\mathcal{F}^2_{\phi(G)}}2^{-R} \in (\frac{1}{4|\zeta^i_u|}, \frac{1}{4}]. \tag{6.22}$$

This together with the way s was determined implies that (6.19d) holds. Notice $s \gtrsim A$, therefore (6.19c) actually follows from (6.2) by multiplying both sides respectively by s^{-1} and A^{-1}. It only remains to check (6.19b), which would follow if

$$2^R \gtrsim \mathcal{M}_\psi A^2 2^{s^{10}+\kappa_7\mathcal{F}^2_{\phi(G)}l} \sim \mathcal{M}_\psi 2^{s^{10}+\kappa_7\mathcal{F}^2_{\phi(G)}l+2d_i\delta T}. \tag{6.23}$$

It suffices to assume

$$\begin{aligned} R \geq & \log \mathcal{M}_\psi + s^{10} + \kappa_7 \mathcal{F}_{\phi(G)}^2 l + 2d_i\delta T \\ \sim & \log \mathcal{M}_\psi + 2^{5d(d-1)\kappa_3\mathcal{F}_{\phi(G)}} \mathcal{M}_\psi^{10d} 2^{(10+10(d+1+\kappa_8\mathcal{F}_{\phi(G)}^2)d_i\delta)T} \\ & + \kappa_7\mathcal{F}_{\phi(G)}^2 2^{d_i\delta T} + 2d_i\delta T \end{aligned} \tag{6.24}$$

Remark (6.24) would also assure that $\mathcal{M}_\psi A|\Delta| l^{\kappa_8\mathcal{F}_{\phi(G)}^2} 2^{-R} \ll 1$ (recall $|\Delta| \leq s^{-3} < 1$), so that the definition of the positive integer n in (6.22) makes sense.

For a sufficiently large constant $\kappa_9 = \kappa_9(d)$, if $T \geq 2$ and

$$R \geq \cdot 2^{\kappa_9\mathcal{F}_{\phi(G)}^2 (T-1)} \mathcal{M}_\psi^{10d} \tag{6.25}$$

then (6.24) holds and therefore so do all the inequalities (6.2) and (6.19a-e), as a consequence (6.18) holds whenever $|\xi| \leq A$, $\xi \neq 0$.

So far we have almost proved the following result:

Proposition 6.6. *We can find effective constants $\kappa_{10}(d)$, $\kappa_{11}(d)$, $\kappa_{12}(d)$ such that:*

Assuming Conditions 2.8 and 1.8; if

$$\log\frac{1}{\epsilon} > \max(\mathcal{M}_\psi^{30d}, 4), \tag{6.26}$$

and

$$\delta \in [\kappa_{10}\mathcal{F}_{\phi(G)}^2 (\log\log\frac{1}{\epsilon})^{-1}, \frac{\alpha}{10}], \tag{6.27}$$

then there exist:

(1) A number $A \geq (\log\frac{1}{\epsilon})^{\kappa_{10}{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\delta}$;

(2) A measure τ' with total mass $|\tau'| \geq \alpha - 5\delta$ which is dominated by some τ'' in the convex hull of $B_G^{\mathrm{Mah}}(\kappa_{11}\log\frac{1}{\epsilon}).\tau := \{\times_{\phi(g)}.\tau | m(g) \leq \kappa_{11}\log\frac{1}{\epsilon}\}$, where $\tau = \psi_*\mu$ and m is the logarithmic Mahler measure;
such that $|\widehat{\tau'}(\xi)|^2 \leq \kappa_{12}A^{-1}$ for any non-trivial character $\xi \in X^*$ with $|\xi| \leq A$, where $|\xi|$ is defined in Definition 3.4.

Proof. As in Proposition 4.16, put $\sqrt{d}2^{-R_0} = \eta = \mathcal{M}_\psi^{-1}\epsilon$. Set

$$T = \lceil \kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log(\mathcal{M}_\psi^{-10d}\delta\log\frac{1}{\epsilon})\rceil. \tag{6.28}$$

We claim Proposition 4.16 applies, to see this it suffices to verify the conditions $\delta \geq \frac{\kappa_4\mathcal{M}_\psi^d}{R_0}$ and $T \in [\frac{1}{\delta}, \frac{\delta R_0}{2}]$.

Set $\kappa_{10} = 3\kappa_9$. When $\log\frac{1}{\epsilon} \geq 4$, $\log\log\frac{1}{\epsilon} \lesssim (\log\frac{1}{\epsilon})^{\frac{1}{3}}$; furthermore we assumed $\mathcal{M}_\psi^{10d} \leq (\log\frac{1}{\epsilon})^{\frac{1}{3}}$ as well. Thus as $\mathcal{F}_{\phi(G)} \gtrsim_d 1$, when $\kappa_9 = \kappa_9(d)$ is large enough we have $\kappa_{10}\mathcal{F}_{\phi(G)}^2 \geq 1$ and

$$\begin{aligned} T \geq & \kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log(\kappa_{10}\mathcal{F}_{\phi(G)}^2\mathcal{M}_\psi^{-10d}\frac{\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}) \\ \geq & \kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log\sqrt[3]{\log\frac{1}{\epsilon}} \\ \geq & \kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\cdot\frac{1}{3}(\kappa_{10}{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\delta)^{-1} \\ = & \delta^{-1}. \end{aligned} \tag{6.29}$$

On the other hand as (6.27) implicitly gives

$$\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log\log\frac{1}{\epsilon}=3\big(\kappa_{10}\mathcal{F}_{\phi(G)}^{2}(\log\log\frac{1}{\epsilon})^{-1}\big)^{-1}\geq 3\delta^{-1}\geq 3,$$

when κ_9 is sufficiently large

$$\begin{aligned}T\leq&\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log(M^{-10d}\delta\log\frac{1}{\epsilon})+1\\ \leq&\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log\log\frac{1}{\epsilon}+1\\ \leq&\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\cdot 2\log\log\frac{1}{\epsilon}\\ \leq&\frac{\log\frac{1}{\epsilon}}{2\log\log\frac{1}{\epsilon}}\leq\frac{\delta\log\frac{1}{\epsilon}}{2}\leq\frac{\delta R_0}{2}.\end{aligned}\tag{6.30}$$

By similar arguments, it is quite easy to deduce $\delta\geq\frac{\kappa_4\mathcal{M}_\psi^d}{R_0}$ using $R_0\geq\log\frac{1}{\epsilon}>\mathcal{M}_\psi^{30d}$ and $\mathcal{F}_{\phi(G)}\gtrsim_d 1$.

Thus we can obtain R, ν and i from Proposition 4.16. Notice $R\geq\delta R_0\geq\delta\log\frac{1}{\epsilon}$, therefore (6.25) follows from (6.28). And we may define the numbers n, s, l, A as in (6.20), (6.21) and (6.22). Then (6.18) was already proved for all non trivial characters $\xi\in X^*$ with $|\xi|\leq A$. Denote the implied constant by κ_{12}, which is effective.

Let τ' be the measure $\nu^i_{n,s,l}$ defined in Notation 6.3. Since τ' is the average of a certain collection of pushforwards of ν, $|\tau'|=|\nu|\geq\alpha-5\delta$. Because $\nu\leq\tau$, τ' is dominated by the probability measure $\tau'':=\frac{1}{sl}\sum_{t=0}^{s-1}\sum_{k=1}^{l}(u^n a_t b_k).\tau$, which lies in the convex hull of

$$\{g.\tau,\ h^{\mathrm{Mah}}(\phi(g))\leq nh^{\mathrm{Mah}}(\phi(u))+\max_t h^{\mathrm{Mah}}(\phi(a_t))+\max_k h^{\mathrm{Mah}}(\phi(b_k))\big)\}$$

inside the space of probability measures on X.

Observe $h^{\mathrm{Mah}}(\phi(u))=\log|\zeta_u^i|$ as V_i is the only expanding subspace under $\times_u$. By Propositions in §5, $nh^{\mathrm{Mah}}(\phi(u))+\max_t h^{\mathrm{Mah}}(\phi(a_t))+\max_k h^{\mathrm{Mah}}(\phi(b_k))\leq n\log|\zeta_u^i|+s^{10}+\kappa_7\mathcal{F}_{\phi(G)}^2 l$, which we will show can be bounded by a multiple of $\log\frac{1}{\epsilon}$.

It follows from (6.22) that $|\zeta_u^i|^n\leq\mathcal{M}_\psi^{-1}|\Delta|^{-1}2^R\leq\mathcal{M}_\psi^{-1}s^{\kappa_6}2^R$ and by (6.24) $s^{10}\mathcal{F}_{\phi(G)}^2 l\leq R-\log\mathcal{M}_\psi$. So $n\log|\zeta_u^i|+s^{10}+\kappa_7\mathcal{F}_{\phi(G)}^2 l\leq(\kappa_6\log s+R-\log\mathcal{M}_\psi)+(R-\log\mathcal{M}_\psi)\lesssim_d s^{10}+R-\log\mathcal{M}_\psi\lesssim R_0-\log\mathcal{M}_\psi=\log\frac{\sqrt{d}}{\epsilon}$. Thus by Lemma 2.10 there exists an effective constant $\kappa_{11}(d)$ such that τ'' is in the the convex hull of $B_G^{\mathrm{Mah}}(\kappa_{11}\log\frac{1}{\epsilon}).\tau$.

It only remains to obtain the lower bound for A. As we already remarked in (6.29), $T\geq\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log\sqrt[3]{\log\frac{1}{\epsilon}}$. Hence $A\geq 2^{d_i\delta T}\geq 2^{\delta\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\log\sqrt[3]{\log\frac{1}{\epsilon}}}=(\log\frac{1}{\epsilon})^{\frac{1}{3}\kappa_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\delta}=(\log\frac{1}{\epsilon})^{\kappa_{10}{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\delta}$. □

Now we may pull everything from X back to $\mathbb{T}^d$. The following corollary is immediate.

Corollary 6.7. *For the same constants κ_{10}, κ_{11} and κ_{12} as in Proposition 6.6, if Conditions 2.8 and 1.8 hold and*

$$\log\frac{1}{\epsilon}>\max(\mathcal{M}_\psi^{30d},4);$$

$$\delta \in [5\kappa_{10}\mathcal{F}^2_{\phi(G)}(\log\log\frac{1}{\epsilon})^{-1}, \frac{\alpha}{2}],$$

then there exist:

(1) A number $A \geq (\log\frac{1}{\epsilon})^{\frac{1}{5}\kappa_{10}{}^{-1}\mathcal{F}^{-2}_{\phi(G)}\delta}$*;*

(2) A measure μ' *with total mass* $|\mu'| \geq \alpha - \delta$ *which is dominated by some* μ'' *in the convex hull of* $B^{\mathrm{Mah}}_G(\kappa_{11}\log\frac{1}{\epsilon}).\mu$*;*

such that $|\widehat{\mu'}(\mathbf{q})|^2 \leq \kappa_{12}A^{-1}$ *for all non-trivial characters* $\mathbf{q} \in \mathbb{Z}^d$ *with* $|\mathbf{q}| \leq A$*.*

Proof. Apply Proposition 6.6 to $\frac{\delta}{5}$ instead of δ and pull back from X back to $\mathbb{T}^d$ by ψ. □

6.3 The effective measure-theoretical theorem

In Corollary 6.7 we constructed a measure μ' whose Fourier coefficients are small except the trivial one. Since for the Lebesgue measure m on $\mathbb{T}^d$, all the Fourier coefficients at non-trivial frequencies vanish. It is natural that we can effectively describe how close μ' is to a multiple of m.

Definition 6.8. *For a function* $f \in \mathcal{C}^\infty(\mathbb{T}^d)$ *and* $w > 0$, $\|f\|_{\dot{H}^w}$ *denotes the Sobolev norm* $\big(\sum_{\mathbf{q}\in\mathbb{Z}^d}\big||\mathbf{q}|^w\hat{f}(\mathbf{q})\big|^2\big)^{\frac{1}{2}}$.

Lemma 6.9. *If a measure* γ *on* $\mathbb{T}^d$ *satisfies* $|\gamma| \leq 1$ *and* $|\hat{\gamma}(\mathbf{q})| \leq CA^{-\frac{1}{2}}$, $\forall \mathbf{q} \in \mathbb{Z}^d\backslash\{0\}, |\mathbf{q}| \leq A$ *for some fixed positive number* C*, then there is an explicit constant* $\kappa_{13} = \kappa_{13}(d)$ *such that* $\forall f \in \mathcal{C}^\infty(\mathbb{T}^d)$*,*

$$\big|\gamma(f) - |\gamma|\mathrm{m}(f)\big| \leq \kappa_{13}\cdot(C+1)A^{-\frac{1}{2}}\|f\|_{\dot{H}^{\frac{d+1}{2}}}.$$

Proof. Denote $B_{\mathbb{Z}^d}(L) = \{\mathbf{q} \in \mathbb{Z}^d, |\mathbf{q}| \leq L\}$. If f is sufficiently differentiable then the spherical Fourier sums

$$S_L(f)(x) := \sum_{\mathbf{q}\in B_{\mathbb{Z}^d}(L)} \hat{f}(\mathbf{q})e(\mathbf{q}\cdot x), x \in \mathbb{T}^d$$

coverge uniformly to f as $L \to \infty$ (see Il'in [11]).

Hence $\gamma(f) = \lim_{L\to\infty}\gamma(S_L(f)) = \lim_{L\to\infty}\sum_{\mathbf{q}\in B_{\mathbb{Z}^d}(L)}\hat{\gamma}(\mathbf{q})\hat{f}(\mathbf{q})$ and

$$\begin{aligned}
&\big|\gamma(f) - |\gamma|\mathrm{m}(f)\big| \\
=&|\gamma(f) - \hat{\gamma}(0)\hat{f}(0)| \\
=&|\lim_{L\to\infty}\sum_{\mathbf{q}\in B_{\mathbb{Z}^d}(L)\backslash\{0\}}\hat{\gamma}(\mathbf{q})\hat{f}(\mathbf{q})| \\
\leq&\overline{\lim}_{L\to\infty}\big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(B_{\mathbb{Z}^d}(L)\backslash\{0\})}\big\||\mathbf{q}|^{\frac{d+1}{2}}\hat{f}(\mathbf{q})\big\|_{l^2(B_{\mathbb{Z}^d}(L)\backslash\{0\})} \qquad (6.31)\\
&\text{(Cauchy-Schwarz)} \\
\leq&\big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(\mathbb{Z}^d\backslash\{0\})}\big\||\mathbf{q}|^{\frac{d+1}{2}}\hat{f}(\mathbf{q})\big\|_{l^2(\mathbb{Z}^d)} \\
\leq&\big(\big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(B_{\mathbb{Z}^d}(A)\backslash\{0\})} + \big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(\mathbb{Z}^d\backslash B_{\mathbb{Z}^d}(A))}\big)\|f\|_{\dot{H}^{\frac{d+1}{2}}}.
\end{aligned}$$

However it follows from the assumption that $\big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(B_{\mathbb{Z}^d}(A)\backslash\{0\})} \leq CA^{-\frac{1}{2}}\big\||\mathbf{q}|^{-\frac{d+1}{2}}\big\|_{l^2(B_{\mathbb{Z}^d}(A)\backslash\{0\})} \leq C\big\||\mathbf{q}|^{-\frac{d+1}{2}}\big\|_{l^2(\mathbb{Z}^d\backslash\{0\})}\cdot A^{-\frac{1}{2}}$. Remark the norm $\big\||\mathbf{q}|^{-\frac{d+1}{2}}\big\|_{l^2(\mathbb{Z}^d\backslash\{0\})}$ is a finite constant relying only on d.

On the other hand, as $\hat{\gamma}(\mathbf{q}) \leq |\gamma| \leq 1$ for any $\mathbf{q}$, $\big\||\mathbf{q}|^{-\frac{d+1}{2}}\hat{\gamma}(\mathbf{q})\big\|_{l^2(\mathbb{Z}^d\setminus B_{\mathbb{Z}^d}(A))}$ is bounded by $\big\||\mathbf{q}|^{-\frac{d+1}{2}}\big\|_{l^2(\mathbb{Z}^d\setminus B_{\mathbb{Z}^d}(A))} \sim_d \|y^{-\frac{d+1}{2}}\|_{L^2(\mathbb{R}^d\setminus B_{\mathbb{R}^n}(A))}$, which has order $(\int_{y\in\mathbb{R}^d,|y|>A} y^{-d-1}\mathrm{d}y)^{\frac{1}{2}} \sim_d (A^{-1})^{\frac{1}{2}} = A^{-\frac{1}{2}}$ where $B_{\mathbb{R}^n}(A)$ denotes the Euclidean ball of radius A in $\mathbb{R}^n$.

By plugging these bounds into (6.31), it is clear that $\big|\gamma(f)-|\gamma|\mathrm{m}(f)\big| \lesssim_d (CA^{-\frac{1}{2}}+A^{-\frac{1}{2}})\|f\|_{\dot{H}^{\frac{d+1}{2}}} = (C+1)A^{-\frac{1}{2}}\|f\|_{\dot{H}^{\frac{d+1}{2}}}$. □

Applying lemma to the measure μ' from Corollary 6.7, we obtain the following:

Proposition 6.10. *There exist effective constants $c_3(d)$, $c_6(d)$ and $c_9(d)$, such that:*

If G and μ satisfy Condition 2.8 and 1.8 respectively, $\log\frac{1}{\epsilon} \geq \max(\mathcal{M}_\psi^{30d}, 4)$ and $\delta \in [c_9\mathcal{F}_{\phi(G)}^2(\log\log\frac{1}{\epsilon})^{-1}, \frac{\alpha}{2}]$, then there exists a measure μ' with total mass $|\mu'| \geq \alpha - \delta$ which is dominated by some μ'' in the convex hull of $B_G^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon}).\mu$ such that

$$\big|\mu'(f) - |\mu'|\mathrm{m}(f)\big| \leq c_6(\log\frac{1}{\epsilon})^{-\frac{1}{2}c_9{}^{-1}\mathcal{F}_{\phi(G)}^{-2}\delta}\|f\|_{\dot{H}^{\frac{d+1}{2}}}, \forall f \in \mathcal{C}^\infty(\mathbb{T}^d).$$

Proof. Let $c_9 = 5\kappa_{10}$, $c_3 = \kappa_{11}$, $c_6 = \kappa_{13}(d)(\sqrt{\kappa_{12}}+1)$ and apply Lemma 6.9 to μ' from Corollary 6.7. Note all constants are effective and depend only on the dimension d. □

Proposition 6.10 trivially implies the measure-theoretical form (Theorem 1.9) of our main result.

Proof of Theorem 1.9. By Theorem 2.12, Condition 2.8 is implied by Condition 1.5, so Theorem 6.10 applies.

Let $c_1 = 2^{-\max(\mathcal{M}_\psi^{30d},4)}$, $c_5 = c_9\mathcal{F}_{\phi(G)}^2$, and c_3, c_6 be the same as in Theorem 6.10. All these constants depend effectively on d, $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$, therefore evenually only on the group G. □

7 Topological results

We now prove the main result of this paper, as well as two corollaries which give a clearer picture of the G-action on the torus.

7.1 Density of the orbit of a dispersed set

In parallel to the measure-theoretical Proposition 6.10, we prove first a topological result which assumes Condition 2.8 and specifies how the constants depend on number-theoretical features of group G.

A subset E of a metric space is said to be ϵ-separated if any pair of points in E are at least of distance ϵ from each other.

Proposition 7.1. *There are effective constants $c_3(d)$, $c_{10}(d)$ and $c_{11}(d)$ such that:*

Suppose $G < SL_d(\mathbb{Z})$ satisfies Condition 2.8, if $\log\frac{1}{\epsilon} \geq \max(\mathcal{M}_\psi^{30d}, 4)$ and some ϵ-separated set E has size $|E| \geq \epsilon^{-\alpha d}$ where $\alpha \in [c_{10}\mathcal{F}_{\phi(G)}^2\frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}, 1)$, then $B_G^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon}).E$ is $(\log\frac{1}{\epsilon})^{-c_{11}\mathcal{F}_{\phi(G)}^{-2}\alpha}$-dense.

Proof. Let $\mu = \frac{1}{|E|}\sum_{x\in E}\delta_x$ be the uniform probability measure on E. For any partition $\mathcal{P}$ of the torus such that diam$\mathcal{P} \le \epsilon$, each atom contains at most one point from E. Thus $H_\mu(\mathcal{P}) = \sum_{P\in\mathcal{P}} -\mu(P)\log\mu(P) = \sum_{x\in E} -\frac{1}{|E|}\log\frac{1}{|E|} = \log|E| \ge \alpha d\log\frac{1}{\epsilon}$ satisfies Condition 1.8.

Fix a positive bump function $\theta \in \mathcal{C}^\infty(\mathbb{R}^d)$ supported on the ball $B(0,1)$ centered at the origin with radius 1 such that $\int_{y\in\mathbb{R}^d}\theta(y)\mathrm{d}y = 1$. For any $x\in\mathbb{T}^d$ and any positive number $\rho < \frac{1}{2}$, on the ball $B(x,\rho)\subset\mathbb{T}^d$ centered at x with radius ρ we define $f(x+v) = \rho^{-d}\theta(\frac{v}{\rho}), v\in\mathbb{R}^d, |v|\le r$; on the rest of $\mathbb{T}^d$ let f be equal to 0. Then f is a positive function in $\mathcal{C}^\infty(\mathbb{T}^d)$ and $\mathrm{m}(f) = 1$.

Remark

$$
\begin{aligned}
\|f\|_{\dot{H}^{\frac{d+1}{2}}} &\lesssim_d \Big(\sum_{\mathbf{q}\in\mathbb{Z}^d}\sum_{j=1}^{d}|q_j|^{d+1}|\hat{f}(\mathbf{q})|^2\Big)^{\frac{1}{2}} \\
&\le \Big(\sum_{j=1}^{d}\sum_{\mathbf{q}\in\mathbb{Z}^d}\big(q_j^{\lceil\frac{d+1}{2}\rceil}\hat{f}(\mathbf{q})\big)^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{j=1}^{d}\|(-\frac{\partial}{\partial x_j})^{\lceil\frac{d+1}{2}\rceil}f\|^2_{L^2(\mathbb{T}^d)}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Moreover, an easy fact is that the last expression is proportional to $\rho^{-\frac{d}{2}-\lceil\frac{d+1}{2}\rceil}$ once θ is fixed. Thus there is an effective dimensional constant $\iota_1(d)$ such that $\|f\|_{\dot{H}^{\frac{d+1}{2}}} \le \iota_1\rho^{-\frac{d}{2}-\lceil\frac{d+1}{2}\rceil}$.

If $\alpha \ge c_{10}\mathcal{F}^2_{\phi(G)}\frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}$ with $c_{10} = \max(8c_9, 3c_6\iota_1{\kappa_1}^{-2})$, which depends effectively on d, then as $\log\log\log\frac{1}{\epsilon} \ge \log\log 4 = 1$, we can apply Proposition 6.10 to μ with $\delta = \frac{\alpha}{2}$ and get a measure μ' such that

$$
\begin{aligned}
\mu'(f) \ge&(\alpha-\delta)\mathrm{m}(f) - c_6(\log\frac{1}{\epsilon})^{-\frac{1}{2}{c_9}^{-1}\mathcal{F}^{-2}_{\phi(G)}\delta}\cdot\iota_1\rho^{-\frac{d}{2}-\lceil\frac{d+1}{2}\rceil} \\
=&\frac{\alpha}{2} - c_6\iota_1(\log\frac{1}{\epsilon})^{-\frac{1}{4}{c_9}^{-1}\mathcal{F}^{-2}_{\phi(G)}\alpha}\cdot\rho^{-\frac{d}{2}-\lceil\frac{d+1}{2}\rceil}.
\end{aligned}
\tag{7.1}
$$

Take

$$
\rho_0 = (\log\frac{1}{\epsilon})^{-\frac{1}{8}(\frac{d}{2}+\lceil\frac{d+1}{2}\rceil)^{-1}{c_9}^{-1}\mathcal{F}^{-2}_{\phi(G)}\alpha}. \tag{7.2}
$$

Then (7.1) rewrites

$$
\begin{aligned}
\mu'(f) \ge&\frac{\alpha}{2} - c_6\iota_1(\log\frac{1}{\epsilon})^{-\frac{1}{8}{c_9}^{-1}\mathcal{F}^{-2}_{\phi(G)}\alpha} \\
\ge&\frac{1}{2}c_{10}\mathcal{F}^2_{\phi(G)}\frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}} - c_6\iota_1(\log\frac{1}{\epsilon})^{-\frac{c_{10}}{8c_9}\cdot\frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}} \\
\ge&\frac{1}{2}c_{10}{\kappa_1}^2\frac{1}{\log\log\frac{1}{\epsilon}} - c_6\iota_1(\log\frac{1}{\epsilon})^{-\frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}}} \\
\ge&(\frac{1}{2}c_{10}{\kappa_1}^2 - c_6\iota_1)\cdot\frac{1}{\log\log\frac{1}{\epsilon}}.
\end{aligned}
\tag{7.3}
$$

So as $c_{10} > 2c_6\iota_1{\kappa_1}^{-2}$, $\mu'(f)$ is strictly positive. In particular, $\mathrm{supp}\mu'\cap\mathrm{supp}f \ne \emptyset$. Since $\mathrm{supp}\mu = E$ and there is μ'' in the convex hull of $B_F^{\mathrm{Mah}}(c_3\log\frac{1}{\epsilon}).\mu$ such that $\mu' \le \mu''$, $\mathrm{supp}\mu' \subset$

$\mathrm{supp}\mu'' \subset B_F^{\mathrm{Mah}}(c_3 \log \frac{1}{\epsilon}).E$. Moreover f is supported on $B(x, \rho_0)$ for an arbitrary point $x \in \mathbb{T}^d$, therefore we proved $B_F^{\mathrm{Mah}}(c_3 \log \frac{1}{\epsilon}).E$ is ρ_0-dense. It suffices to let c_3 be the same as in Proposition 6.10 and set $c_{11} = \frac{1}{8}(\frac{d}{2} + \lceil \frac{d+1}{2} \rceil)^{-1} c_9{}^{-1}$ to conclude. □

Proof of Theorem 1.7. By Theorem 2.12, Proposition 7.1 applies.

Let $c_1 = 2^{-\max(\mathcal{M}_\psi^{30d}, 4)}$, c_3 be the same as in Theorem 6.10, $c_2 = c_{10}\mathcal{F}_{\phi(G)}^2$ and $c_4 = c_{11}\mathcal{F}_{\phi(G)}^{-2}$; all of which are effective and depend on d, $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$, thus eventually only on G itself. □

7.2 Density of the orbit of a large set

In Theorems 7.1 and 1.7, we assumed the set E is sufficiently dispersed with respect to its size. As a corollary to these theorems, the results in this subsection drop this hypothesis. We are going to show that the full G-orbit of any sufficiently large subset E in $\mathbb{T}^d$ is going to have some density.

If we care about a metric ball $B_G^{\mathrm{Mah}}(L).E$ with respect to the logarithmic Mahler measure (or the word metric) on G in the orbit instead to the full orbit, then it is still necessary to assume E is ϵ_0-separated for certain ϵ_0. Nevertheless, ϵ_0 is not going to rely on $|E|$. Moreover, ϵ_0 only affects how large the metric ball has to be but has no impact on the effective density we obtain.

Proposition 7.2. *Suppose G satisifes Condition 2.8 and the image $\psi(E)$ of a finite set $E \subset \mathbb{T}^d$ under ψ is $\epsilon_0\mathcal{S}_\Gamma$-separated in $X = \psi(\mathbb{T}^d)$. If*

$$\log\log|E| \geq \max\left(c_{12}\mathcal{M}_\psi^{30d}, \max(\mathcal{F}_{\phi(G)}, 2)^{c_{13}\mathcal{F}_{\phi(G)}^2}\right), \tag{7.4}$$

then the set $B_G^{\mathrm{Mah}}(\log\frac{1}{\epsilon_0} + c_{14}\log|E|).E$ is $(\log\log|E|)^{-c_{15}\mathcal{F}_{\phi(G)}^2}$-dense, where $c_{12}, \cdots, c_{15}$ are effective constants that depend only on d.

Remark 7.3. *Here we adopt the assumption that $\psi(E)$ is $\epsilon_0\mathcal{S}_\Gamma$ dense in X because this would be the more convenient formulation for later applications. However it is completely fine to replace the assumption by a more natural one, namely E is ϵ_0-separated in $\mathbb{T}^d$. Actually if this holds then $\psi(E)$ is $\|\psi^{-1}\|^{-1}\epsilon_0$-separated in X. However $\|\psi^{-1}\|^{-1}\epsilon_0 \geq \mathcal{M}_\psi^{-1}\epsilon_0\mathcal{S}_\Gamma$. So the proposition applies with $\mathcal{M}_\psi^{-1}\epsilon_0$ in place of ϵ_0 and concludes that the set $B_G^{\mathrm{Mah}}(\log\frac{1}{\epsilon_0} + \log\mathcal{M}_\psi + c_{14}\log|E|).E$ is $(\log\log|E|)^{-c_{15}\mathcal{F}_{\phi(G)}^2}$-dense. However since $\log|E| \gg \log\log|E| \gtrsim_d \mathcal{M}_\psi^{30d} \gg \log\mathcal{M}_\psi$, by making c_{14} slightly larger the term $\log\mathcal{M}_\psi$ can be absorbed into $c_{14}\log|E|$ and the original claim still holds.*

Proof of Proposition. As the torus $\mathbb{T}^d$ can be covered by at most $|E|-1$ balls of radius $\frac{\sqrt{d}}{\lfloor \sqrt[d]{|E|-1} \rfloor} \lesssim_d |E|^{-\frac{1}{d}}$, there are two elements $z, z' \in E$ such that $|z - z'| \lesssim_d |E|^{-\frac{1}{d}}$, where $|z - z'|$ denotes the distance in $\mathbb{T}^d$

We select lifts $\tilde{z}, \tilde{z}' \in \mathbb{R}^d$ of $z, z' \in \mathbb{T}^d$ such that $|\tilde{z} - \tilde{z}'| = |z - z'|$. Denote $\tilde{y} = \psi(\tilde{z} - \tilde{z}') \in \mathbb{R}^d$, then by the fact $\|\psi\| \leq \mathcal{M}_\psi\mathcal{S}_\Gamma$, $|\tilde{y}| \leq \iota_1\mathcal{M}_\psi|E|^{-\frac{1}{d}}\mathcal{S}_\Gamma$, where ι_1 depends effectively on d. On the other hand $\tilde{y}$ is a lift of $\psi(z) - \psi(z')$, thus $|\tilde{y}| \geq \epsilon_0\mathcal{S}_\Gamma$ by assumption.

$\tilde{y}$ can be decomposed as $\sum_{j=1}^{r_1+r_2} \tilde{y}_j$ where $\tilde{y}_j$ is the orthogonal projection onto V_j. Without loss of generality, assume $|\tilde{y}_i| \geq |\tilde{y}_j|, \forall j \neq i$; then

$$|\tilde{y}_i| \geq (r_1 + r_2)^{-\frac{1}{2}}\epsilon_0 S_\Gamma \tag{7.5}$$

but

$$|\tilde{y}_j| \leq |\tilde{y}_i| \leq \iota_1\mathcal{M}_\psi|E|^{-\frac{1}{d}}\mathcal{S}_\Gamma, \forall j \neq i. \tag{7.6}$$

Let s be the largest positive integer such that

$$2^{s^{10}}\iota_1|E|^{-\frac{1}{3d}} \leq \frac{1}{4}(r_1+r_2)^{-\frac{1}{2}}, \tag{7.7}$$

then by the fact $\mathcal{F}_\phi(G) \gtrsim_d 1$, for some effective constant $\iota_2(d)$ under the assumption

$$\log|E| \geq \iota_2 \mathcal{F}_{\phi(G)}^{10} \tag{7.8}$$

it is guaranteed that

$$s \sim_d (\log|E|)^{\frac{1}{10}} \tag{7.9}$$

and

$$s \geq \max\left(\kappa_5 \mathcal{F}_{\phi(G)}, 4\right),$$

where κ_5 was given by Proposition 5.5.

Therefore we may apply Propositon 5.5 to construct a sequence of elements $a_0, \cdots, a_{s-1} \in G$. We are interested in the set $\{\times_{\phi(a_t)}.\tilde{y}\}_{t=0,\cdots,s-1}$. Notice the j-th coordinate of the t-th element is $\zeta_{a_t}^j \tilde{y}_j$.

By Propositon 5.5, $|\zeta_{a_t}^i - (1+t\Delta)| \leq s^{-1}|\Delta| \leq \frac{1}{4}|\Delta|$. Thus for any two distinct t, t', $|\zeta_{a_t}^i \tilde{y}_i - \zeta_{a_{t'}}^i \tilde{y}_i|$ is bounded from below by $(|\Delta| - 2\cdot\frac{1}{4}|\Delta|)y_i \geq \frac{|\Delta|}{2}|y_i|$, and from above by $\left((s-1)|\Delta| + 2\cdot\frac{1}{4}|\Delta|\right)|\tilde{y}_i| \leq s|\Delta|\cdot|\tilde{y}_i|$, where $s^{-\kappa_6\mathcal{F}_{\phi(G)}^2} \leq |\Delta| \leq s^{-3}$.

Assume in addition

$$|E| \geq \mathcal{M}_\psi^{3d}, \tag{7.10}$$

then $\mathcal{M}_\psi|E|^{-\frac{1}{d}} \leq \mathcal{M}_\psi^{-1}|E|^{-\frac{1}{3d}}$.

So as long as (7.8) holds with sufficiently large ι_2, as $\mathcal{F}_{\phi(G)} \gtrsim_d 1$ and $r_1 + r_2 \leq d-1$,

$$\begin{aligned} s|\Delta|\cdot|\tilde{y}_i| \leq& s\cdot s^{-3}\cdot\iota_1\mathcal{M}_\psi|E|^{-\frac{1}{d}}\mathcal{S}_\Gamma \\ \leq& \iota_1\mathcal{M}_\psi^{-1}|E|^{-\frac{1}{3d}}\mathcal{S}_\Gamma \\ \leq& \frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma. \end{aligned}$$

Let $u \in G$ be given by Proposition 5.2, then $|\zeta_u^i| \geq 2^{d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}$ and $|\zeta_u^j| < 1$ for all $j \neq i$. By the observation above, we may take an integer $n \geq 0$ such that

$$|\zeta_u^i|^{-1}\cdot\frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma \leq |\zeta_u^i|^n s|\Delta|\cdot|\tilde{y}_i| \leq \frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma, \tag{7.11}$$

which this also implies

$$|\zeta_u^i|^n\cdot s|\Delta|\cdot|\tilde{y}_i| \geq |\zeta_u^i|^{-1}\cdot\frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma. \tag{7.12}$$

For any pair $t \neq t'$,

$$\begin{aligned} |\zeta_{u^n a_t}^i\tilde{y}_i - \zeta_{u^n a_{t'}}^i\tilde{y}_i| =& |\zeta_u^i|^n|\zeta_{a_t}^i\tilde{y}_i - \zeta_{a_{t'}}^i\tilde{y}_i| \leq |\zeta_u^i|^n\cdot s|\Delta|\cdot|\tilde{y}_i| \\ \leq& \frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma. \end{aligned} \tag{7.13}$$

On the other hand, by Propositon 5.5, $h^{\mathrm{Mah}}(\phi(a_t)) \leq s^{10}$, hence $|\zeta^j_{a_t}| \leq 2^{s^{10}}, \forall j$. Therefore if $j \neq i$, using $|\zeta^j_u| < 1$ and (7.6), (7.7), (7.10),

$$\begin{aligned} &|\zeta^j_{u^n a_t}\tilde{y}_j - \zeta^j_{u^n a_{t'}}\tilde{y}_j| \\ \leq &|\zeta^j_u|^n(|\zeta^j_{a_t}| + |\zeta^j_{a_{t'}}|)|\tilde{y}_j| \leq 1 \cdot (2^{s^{10}} + 2^{s^{10}})\iota_1 \mathcal{M}_\psi |E|^{-\frac{1}{d}}\mathcal{S}_\Gamma \\ \leq &2 \cdot 2^{s^{10}}\iota_1 \mathcal{M}_\psi^{-1}|E|^{-\frac{1}{3d}}\mathcal{S}_\Gamma \leq 2 \cdot \frac{1}{4}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma \\ = &\frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma. \end{aligned} \tag{7.14}$$

By (7.13) and(7.14), the orthogonal projection of $\times_{\phi(u^n a_t)}.\tilde{y} - \times_{\phi(u^n a_{t'})}.\tilde{y}$ is bounded by $\frac{1}{2}(r_1 + r_2)^{-\frac{1}{2}}$ in each V_j, $1 \leq j \leq r_1 + r_2$, hence

$$|\times_{\phi(u^n a_t)}.\tilde{y} - \times_{\phi(u^n a_{t'})}.\tilde{y}| \leq \frac{1}{2}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma.$$

Since $\times_{\phi(u^n a_t)}.\tilde{y} = \psi\big(u^n a_t.(\tilde{z} - \tilde{z}')\big), \forall t$ and $\|\psi^{-1}\| \leq \mathcal{M}_\psi \mathcal{S}_\Gamma^{-1}$,

$$|u^n a_t.(\tilde{z} - \tilde{z}') - u^n a_{t'}.(\tilde{z} - \tilde{z}')| \leq \|\psi^{-1}\| \cdot |\times_{\phi(u^n a_t)}.\tilde{y} - \times_{\phi(u^n a_{t'})}.\tilde{y}| \leq \frac{1}{2}.$$

Because $u^n a_t.(\tilde{z} - \tilde{z}') \in \mathbb{R}^d$ is the lift of $u^n a_t.(z - z') \in \mathbb{T}^d$ for all t, this bound implies the distance $|u^n a_t.(z - z') - u^n a_{t'}.(z - z')|$ between $u^n a_t.(z - z')$ and $u^n a_{t'}.(z - z')$ on $\mathbb{T}^d$ is exactly $|u^n a_t.(\tilde{z} - \tilde{z}') - u^n a_{t'}.(\tilde{z} - \tilde{z}')|$. Therefore

$$\begin{aligned} &|u^n a_t.(z - z') - u^n a_{t'}.(z - z')| \\ = &|u^n a_t.(\tilde{z} - \tilde{z}') - u^n a_{t'}.(\tilde{z} - \tilde{z}')| \\ \geq &\|\psi\|^{-1}|\times_{\phi(u^n a_t)}.\tilde{y} - \times_{\phi(u^n a_{t'})}.\tilde{y}| \\ \geq &\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma^{-1}|\times_{\phi(u^n a_t)}.\tilde{y}_i - \times_{\phi(u^n a_{t'})}.\tilde{y}_i| \\ \geq &\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma^{-1}|\zeta^i_u|^n|\times_{\phi(a_t)}.\tilde{y}_i - \times_{\phi(a_{t'})}.\tilde{y}_i| \\ \geq &\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma^{-1}|\zeta^i_u|^n \cdot \frac{|\Delta|}{2}|\tilde{y}_i|. \end{aligned} \tag{7.15}$$

By (7.12),

$$\begin{aligned} (7.15) \geq &\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma^{-1} \cdot \frac{1}{2}s^{-1} \cdot |\zeta^i_u|^{-1} \cdot \frac{1}{2}(r_1+r_2)^{-\frac{1}{2}}\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma \\ \geq &\frac{1}{4}(r_1+r_2)^{-\frac{1}{2}}2^{-h^{\mathrm{Mah}}(\phi(u))}\mathcal{M}_\psi^{-2}s^{-1} \\ \geq &\frac{1}{4}(d-1)^{-\frac{1}{2}}2^{-9d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}}\mathcal{M}_\psi^{-2}s^{-1} \end{aligned}$$

By (7.9), $s \geq \iota_3(\log|E|)^{\frac{1}{10}}$ for an effective $\iota_3 = \iota_3(d)$, so under a new assumption

$$\log|E| \geq \max(2^{c_{14}\mathcal{F}_{\phi(G)}}, \mathcal{M}_\psi^{30}) \tag{7.16}$$

where a sufficiently large effective constant $c_{14} = c_{14}(d)$ is used, we have $\mathcal{M}_\psi^2 < {\iota_3}^{-\frac{2}{3}}s^{\frac{2}{3}}$ and $4(d-1)^{\frac{1}{2}}2^{9d(\frac{r}{2}+1)\mathcal{F}_{\phi(G)}} < \frac{1}{2}\iota_3^{\frac{2}{3}}s^{\frac{1}{3}}$. Thus

$$(7.15) \geq {\iota_3}^{\frac{2}{3}}s^{-\frac{2}{3}} \cdot 2{\iota_3}^{-\frac{2}{3}}s^{-\frac{1}{3}} \cdot s^{-1} \geq 2s^{-2}, \tag{7.17}$$

which can be interpreted as the set $E_1 := \{u^n a_t.(z-z')|0 \le t \le s-1\}$ is $2s^{-2}$-separated in $\mathbb{T}^d$.

Let $E_2 = \{u^n a_t.x | 0 \le t \le s-1, x \in E\} \subset \mathbb{T}^d$ and observe

$$E_1 \subset E_2 - E_2 := \{w - w' | w, w' \in E_2\}.$$

Take a maximal s^{-2}-separated subset E_3 of E_2, we claim $|E_3| \ge \sqrt{s}$. Actually, E_2 is covered by the union of $|E_3|$ balls $\cup_{v \in E_3} B_v$, where B_v is centered at v with radius s^{-2}. Then the set $E_1 \subset E_2 - E_2$ is covered by $\cup_{v,v' \in E}(B_v - B_{v'})$. But for each pair v, v', the set $B_v - B'_v$ is a ball of radius $2s^{-2}$ centerd at $v - v'$, hence contains at most one point from E_1. So the number of pairs, which equals $|E_3|^2$, is at least $|E_1| = s$. This proves the claim.

We want to apply Proposition 7.1 to $\epsilon := s^{-2}$ and $|E_3| \ge s^{\frac{1}{2}} = \epsilon^{-\frac{1}{4}} = \epsilon^{-\alpha d}$ where $\alpha = \frac{1}{4d}$, for which purpose it is necessary to have $\log \frac{1}{\epsilon} \ge \max(\mathcal{M}_\psi^{30d}, 4)$ and $c_{10}\mathcal{F}^2_{\phi(G)} \frac{\log\log\log\frac{1}{\epsilon}}{\log\log\frac{1}{\epsilon}} \le \frac{1}{4d}$. Because

$$\frac{1}{\epsilon} = s^2 \ge {\iota_3}^2 (\log |E|)^{\frac{1}{5}} \tag{7.18}$$

and we assumed $|E|$ is sufficiently large with respect to d, these conditions would respectively follow from

$$\log\log |E| \ge c_{12} \max(\mathcal{M}_\psi^{30d}, 4); \tag{7.19}$$

and

$$\log\log\log |E| \ge c_{13} \mathcal{F}^2_{\phi(G)} \max(\log \mathcal{F}_{\phi(G)}, 1), \tag{7.20}$$

if c_{12} and c_{13} are sufficiently large constants that depend effectively on d.

Again by $\mathcal{F}_{\phi(G)} \gtrsim_d 1$, if c_{13} is large enough with respect to c_{12} then (7.20) implies the $c_{12} \cdot 4$ part in (7.19). So we can combine the two inequalities into assumption (7.4) in the statement of theorem.

Furthermore, notice that (7.4) actually contains all the earlier assumptions (7.8), (7.10) and (7.16) given that c_{12} and c_{13} are large with respect to all previous constants.

So eventually we can apply Proposition 7.1 to claim E_3 is $(\log \frac{1}{\epsilon})^{-\frac{1}{4d} c_{11} \mathcal{F}^2_{\phi(G)}}$-dense. By (7.18) when $|E|$ is sufficiently large as in our situation, $(\log \frac{1}{\epsilon})^{-\frac{1}{4d} c_{11} \mathcal{F}^2_{\phi(G)}} \le (\log\log |E|)^{-c_{15}\mathcal{F}^2_{\phi(G)}}$ for some effective constant $c_{15} = c_{15}(d)$.

Thus E_3 is $(\log\log |E|)^{-c_{15}\mathcal{F}^2_{\phi(G)}}$-dense.

Observe

$$E_3 \subset E_2 = \{u^n a_t | 0 \le t \le s-1\}.E.$$

It is clear that

$$h^{\mathrm{Mah}}(\phi(a_t)) \le s^{10} \sim_d \left((\log |E|)^{\frac{1}{10}}\right)^{10} = \log |E|. \tag{7.21}$$

Now we investigate how large u^n can be.

By (7.5) and (7.11), $|\zeta_u^i|^n \le \epsilon_0^{-1} \cdot s^{-1}|\Delta|^{-1} \cdot \frac{1}{2}\mathcal{M}_\psi^{-1} \le \epsilon_0^{-1} s^{\kappa_6 \mathcal{F}^2_{\phi(G)} - 1}$ as $\mathcal{M}_\psi \ge 1$ and $|\Delta| \ge s^{-\kappa_6 \mathcal{F}^2_{\phi(G)}}$. Because $|\zeta_u^i| > 1$ but $|\zeta_u^j| < 1, \forall j \ne i$ the logarithmic Mahler measure $h^{\mathrm{Mah}}(\phi(u))$ equals $\log |\zeta_u^i|$, so

$$\begin{aligned} h^{\mathrm{Mah}}(\phi(u^n)) =& n h^{\mathrm{Mah}}(\phi(u)) \le n \log |\zeta_u^i| \le \log(\epsilon_0^{-1} s^{\kappa_6 \mathcal{F}^2_{\phi(G)} - 1}) \\ \le& \log \frac{1}{\epsilon_0} + \kappa_6 \mathcal{F}^2_{\phi(G)} \log s. \end{aligned} \tag{7.22}$$

Remark $\log s \lesssim_d \log\log|E|$ and by (7.4), $\log\log|E| \gg \mathcal{F}^2_{\phi(G)}$; thus $\kappa_6 \mathcal{F}^2_{\phi(G)} \log s \ll \kappa_6 (\log\log|E|)^2$. Therefore (7.21) and (7.22) give a bound to the logarithmic Mahler measure of $u^n a_t, \forall t = 0, \cdots, s-1$:

$$m(u^n a_t) = h^{\mathrm{Mah}}(\phi(u^n a_t)) \leq \log \frac{1}{\epsilon_0} + c_{14} \log|E|.$$

So $E_3 \subset B^{\mathrm{Mah}}_G(\log \frac{1}{\epsilon_0} + c_{14}\log|E|).E$ for some effective constant $c_{14}(d)$, which proves the result. $\square$

Theorem 7.4. *Suppose G satisies Condition 1.5. Then there are effective constants c_{14}, c_{16}, c_{17}, which depend only on the group G such that $\forall \epsilon_0$, for any ϵ_0-separated subset $E \subset \mathbb{T}^d$ of size $|E| \geq c_{16}$, the set $B^{\mathrm{Mah}}_G(\log \frac{1}{\epsilon_0} + c_{14}\log|E|).E$ is $(\log\log|E|)^{-c_{17}}$-dense.*

Proof. The theorem follows from Theorem 2.12, Proposition 7.2 (together with Remark 7.3) and the fact that d, $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$ are determined by G. $\square$

In particular, we have the following corollary:

Corollary 7.5. *If Condition 1.5, any G-invariant finite subset $E \subset \mathbb{T}^d$ of size $|E| \geq c_{16}$ is $(\log\log|E|)^{-c_{17}}$-dense where $c_{16} = c_{16}(G)$, $c_{17} = c_{17}(G)$.*

Remark if E contains an irrational point, then it is dense in $\mathbb{T}^d$ by Berend [2]. So the corollary is meaningful only for rational subsets.

7.3 Density of the orbit of a single point

We now study a single orbit instead of the orbit of a set.

Proposition 7.6. *Let G be as in Condition 2.8 and $Q \in \mathbb{N}$ be such that*

$$\log\log\log Q \geq \max\left(c_{18}\mathcal{M}^{30d}_\psi, \max(\mathcal{F}_{\phi(G)}, 2)^{c_{19}\mathcal{F}^2_{\phi(G)}}\right). \tag{7.23}$$

If a point $x \in \mathbb{T}^d$ satisfies either condition (i) or (ii) in Proposition 1.10, Then the set $B^{\mathrm{Mah}}_G\big((k+2)\log Q\big).x$ is $(\log\log\log Q)^{-c_{20}\mathcal{F}^2_{\phi(G)}}$-dense. Here c_{18}, c_{19}, c_{20} are effective constants that depend only on d .

Before proving the theorem, we construct a metric ball inside the orbit $G.x$ which is large in cardinality and satisfies, although quite weak, some separatedness condition.

Lemma 7.7. *Assuming Condition 2.8, for the element $g \in G$ and the constant $\kappa_3 = \kappa_3(d)$ in Proposition 3.3, if $x \in \mathbb{T}^d$ meets one of the conditions in Proposition 1.10 then $\forall m \neq m'$, $0 \leq m, m' < \mathcal{F}^{-1}_{\phi(G)} \log(2^{-d}Q)$, the distance $|\psi(g^m.x) - \psi(g^{m'}.x)|$ between $\psi(g^m.x)$ and $\psi(g^{m'}.x)$ is at least $\mathcal{M}^{-1}_\psi Q^{-k}\mathcal{S}_\Gamma$, where $k = 1$ if x satisifes condition (ii) in Theorem 1.10.*

Proof. We may assume $m > m'$. Let $\tilde{x} \in \mathbb{R}^d$ be a lift of x in $\mathbb{T}^d$. By definition of $\mathcal{F}_{\phi(G)}$, we may take a toral automorphism g from G with $0 < h^{\mathrm{Mah}}(\phi(g)) \leq \mathcal{F}_{\phi(G)}$. The distance $|\psi(g^m.x) - \psi(g^{m'}.x)|$ in $\mathbb{T}^d$ is given by $|\psi\big((g^m - g^{m'}).\tilde{x} - \Xi\big)|$ where Ξ is some vector from $\mathbb{Z}^d$.

As $h^{\mathrm{Mah}}(\phi(g)) > 0$, $g^{m-m'} - \mathrm{id}$ is an invertible matrix thus so is $g^m - g^{m'}$. (Otherwise one of the eigenvalues $\zeta^i_g = \sigma_i(\phi(g))$ of g is a root of unity, so $h^{\mathrm{Mah}}(\phi(g)) = h^{\mathrm{Mah}}(\zeta^i_g) = 0$.)

Remark $\tilde{x} - (g^m - g^{m'})^{-1}\Xi = (g^m - g^{m'})^{-1}.\big((g^m - g^{m'}).\tilde{x} - \Xi\big)$. Hence since $(g^m - g^{m'})^{-1}$ is simultaneously diagonalizable with g over $\mathbb{C}$,

$$\psi(\tilde{x} - (g^m - g^{m'})^{-1}\Xi) = \times_{\phi\big((g^m - g^{m'})^{-1}\big)}.\psi\big((g^m - g^{m'}).\tilde{x} - \Xi\big).$$

Because $\times_{\phi\big((g^m - g^{m'})^{-1}\big)}$ acts as a multiplication on each component of $\mathbb{R}^d = \oplus_{i=1}^{r_1+r_2} V_i$, we see

$$\begin{aligned}
&\tilde{x} - (g^m - g^{m'})^{-1}\Xi \\
\leq&\|\psi^{-1}\| \cdot \big|\psi\big(\tilde{x} - (g^m - g^{m'})^{-1}\Xi\big)\big| \\
\leq&\mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} \max_{1\leq i\leq d} \big(\big|(\zeta_g^i)^m - (\zeta_g^i)^{m'}\big|^{-1}\big)\big|\psi\big((g^m - g^{m'}).\tilde{x} - \Xi\big)\big| \\
=&\mathcal{M}_\psi \mathcal{S}_\Gamma^{-1} \max_{1\leq i\leq d} \big(\big|(\zeta_g^i)^m - (\zeta_g^i)^{m'}\big|^{-1}\big)\big|\psi(g^m.x) - \psi(g^{m'}.x)\big|.
\end{aligned} \tag{7.24}$$

It follows from the inequality $\max(|z-1|, 1) \leq 2\max(|z|, 1), \forall z \in \mathbb{C}$ and the fact $\prod_{1\leq i\leq d} \max(|\zeta_g^i|, 1) = 2^{h^{\mathrm{Mah}}(\phi(g))} \leq 2^{\mathcal{F}_{\phi(G)}}$ that

$$\begin{aligned}
&|\det(g^m - g^{m'})| \\
=&\prod_{1\leq i\leq d} \big(|(\zeta_g^i)^{m-m'} - 1| \cdot |\zeta_g^i|^{m'}\big) \\
\leq&\prod_{1\leq i\leq d} \Big(\max\big(|(\zeta_g^i)^{m-m'} - 1|, 1\big) \cdot \max(|\zeta_g^i|^{m'}, 1)\Big) \\
\leq&2^d \prod_{1\leq i\leq d} \big(\max(|\zeta_g^i|^{m-m'}, 1) \cdot \max(|\zeta_g^i|^{m'}, 1)\big) \\
=&2^d \prod_{1\leq i\leq d} \max(|\zeta_g^i|, 1)^m \leq 2^{d+m\mathcal{F}_{\phi(G)}} < Q.
\end{aligned} \tag{7.25}$$

(i) Suppose x is diophantine generic.

Since $\det(g^m - g^{m'}) \cdot (g^m - g^{m'})^{-1} \in M_d(\mathbb{Z})$, $(g^m - g^{m'})^{-1}\Xi$ can be written in the form $\frac{v}{q}$ where $v \in \mathbb{Z}^d$ and $q \in \mathbb{N}$ is a factor of $\det(g^m - g^{m'})$; in particular, $q \leq Q$. Because $\tilde{x}$ satisfies the same diophantine property as x, Thus by (7.24), there exists $1 \leq j \leq d$ such that

$$\mathcal{M}_\psi \mathcal{S}_\Gamma^{-1}|(\zeta_g^j)^m - (\zeta_g^j)^{m'}|^{-1}|\psi(g^m.x) - \psi(g^{m'}.x)| \geq \prod_{1\leq i\leq d} |(\zeta_g^i)^m - (\zeta_g^i)^{m'}|^{-k},$$

therefore

$$
\begin{aligned}
&|\psi(g^m.x)-\psi(g^{m'}.x)|\\
\geq&\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma|(\zeta_g^j)^m-(\zeta_g^j)^{m'}|\cdot\prod_{1\leq i\leq d}|(\zeta_g^i)^m-(\zeta_g^i)^{m'}|^{-k}\\
=&\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma|(\zeta_g^j)^m-(\zeta_g^j)^{m'}|^{-(k-1)}\cdot\prod_{1\leq i\leq d,i\neq j}|(\zeta_g^i)^m-(\zeta_g^i)^{m'}|^{-k}\\
\geq&\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma\prod_{1\leq i\leq d}\max(|(\zeta_g^i)^m-(\zeta_g^i)^{m'}|,1)^{-k}\\
\geq&\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma\prod_{1\leq i\leq d}\Big(\max(|(\zeta_g^i)^{m-m'}-1|,1)^{-k}\cdot\max(|(\zeta_g^i)|^{m'},1)^{-k}\Big)\\
\geq&\mathcal{M}_\psi^{-1}\mathcal{S}_\Gamma Q^{-k},
\end{aligned}
\tag{7.26}
$$

where the last step was deduced as in (7.25). This completes the proof in case (i).

(ii) If x is rational with denominator Q, then $g^m.\tilde{x}-g^{m'}.\tilde{x}-\Xi$ has denominator Q as well, thus $|g^m.\tilde{x}-g^{m'}.\tilde{x}-\Xi|\geq Q^{-1}$ as long as it doesn't vanish. Since $\|\psi^{-1}\|\leq\mathcal{M}_\psi\mathcal{S}_\Gamma^{-1}$ by definition, $|\psi(g^m.x)-\psi(g^{m'}.x)|=|\psi\big(g^m.\tilde{x}-g^{m'}.\tilde{x}-\Xi\big)|\geq M_\psi^{-1}\mathcal{S}_\Gamma Q^{-1}$. Therefore if suffices to show $g^m.\tilde{x}-g^{m'}.\tilde{x}-\Xi\neq 0$.

Suppose by absurd that $g^m.\tilde{x}-g^{m'}.\tilde{x}-\Xi=0$, then $\tilde{x}=(g^m-g^{m'})^{-1}\Xi$. As we already found out in case (i), the vector $(g^m-g^{m'})^{-1}\Xi$ is rational and its denominator is at most $|\det(g^m-g^{m'})|<Q$. But $\tilde{x}$ has denominator Q, contradiction. $\square$

Proof of Proposition 7.6. Fix $g\in G$ with $0<h^{\mathrm{Mah}}(\phi(g))\leq\mathcal{F}_{\phi(G)}$, which is possible by the definition of $\mathcal{F}_{\phi(G)}$. By lemma, if we denote

$$
E:=\{g^m.x|0\leq m<\min(\mathcal{F}_{\phi(G)}^{-1},1)\log(2^{-d}Q)\},
\tag{7.27}
$$

then the $g^m.x$'s are all different for different m's and $\psi(E)$ is $\mathcal{M}_\psi^{-1}Q^{-k}\mathcal{S}_\Gamma$-separated.

By assumption (7.23), when c_{18} and c_{19} are sufficiently large with respect to d, we can safely claim that $\min(\mathcal{F}_{\phi(G)}^{-1},1)\log(2^{-d}Q)\geq(\log Q)^{\frac{1}{2}}$. Hence

$$
\log\log|E|\gtrsim_d\log\log\log Q\geq\big(c_{18}\mathcal{M}_\psi^{30d},\max(\mathcal{F}_{\phi(G)},2)^{c_{19}\mathcal{F}_{\phi(G)}^2}\big).
$$

By choosing sufficiently large (but still effective) constants c_{18} and c_{19}, this implies assumption (7.4) (using again $\mathcal{F}_{\phi(G)}\gtrsim_d 1$).

Therefore, by Proposition 7.2, the set $B_G^{\mathrm{Mah}}\big(\log(\mathcal{M}_\psi Q^k)+c_{14}\log|E|\big).E$ is $(\log\log|E|)^{-c_{15}\mathcal{F}_{\phi(G)}^2}$-dense.

However, by the definition of E, it is contained in

$$
B_G^{\mathrm{Mah}}\big(\mathcal{F}_{\phi(G)}^{-1}\log(2^{-d}Q)\cdot h^{\mathrm{Mah}}(\phi(g))\big).x.
$$

But $h^{\mathrm{Mah}}(\phi(g))\leq\mathcal{F}_{\phi(G)}$, so $E\subset B_G^{\mathrm{Mah}}\big(\log(2^{-d}Q)\big).x$ and

$$
\begin{aligned}
&B_G^{\mathrm{Mah}}\big(\log(\mathcal{M}_\psi Q^k)+c_{14}\log|E|\big).E\\
\subset&B_G^{\mathrm{Mah}}\big(\log(\mathcal{M}_\psi Q^k)+c_{14}\log|E|+\log(2^{-d}Q)\big).x
\end{aligned}
$$

Notice $|E| \leq \log(2^{-d}Q) + 1 \leq \log Q$ and by (7.23), $\log \mathcal{M}_\psi \lesssim_d \log\log\log Q$. So $9\log\mathcal{M}_\psi + c_{14}\log|E| + \log(2^{-d}Q) \leq 2\log Q$ when Q is sufficiently large (which is true if c_{18} and c_{19} are large.) Thus

$$9\log(\mathcal{M}_\psi Q^k) + c_{14}\log|E| + \log(2^{-d}Q) \leq \log(Q^k) + 2\log Q \leq (k+2)\log Q$$

and $B_G^{\mathrm{Mah}}\big(\log(\mathcal{M}_\psi Q^k) + c_{14}\log|E|\big).E \subset B_G^{\mathrm{Mah}}\big((k+2)\log Q\big).x$.

On the other hand, $(\log\log|E|)^{-c_{15}\mathcal{F}^2_{\phi(G)}} \leq (\log\log\log Q)^{-c_{20}\mathcal{F}^2_{\phi(G)}}$ for $c_{20} = \frac{1}{2}c_{15}$ when c_{18}, c_{19} are large (so that $Q \gg 1$) because $\log\log|E| \gtrsim_d \log\log\log Q$.

Hence we finally proved $B_G^{\mathrm{Mah}}\big((k+2)\log Q\big).x$ is $(\log\log\log Q)^{-c_{20}\mathcal{F}^2_{\phi(G)}}$ dense when c_{18}, c_{19} and c_{20} are properly chosen. □

Proof of Theorem 1.10. By Theorem 2.12, Proposition 7.6 contains the theorem as d, $\mathcal{M}_\psi$ and $\mathcal{F}_{\phi(G)}$ are all determined by the group G. □

8 Appendix: A number-theoretical application

Before finishing the paper, we give an example of how our results can be applied to number theory.

This appendix follows observations by Cerri [5] and Bourgain-Lindenstrauss [4, §0].

Let K be a number field and $\mathcal{O}_K$ be its ring of integers. For an integral ideal I and an invertible residue class $\beta \in (\mathcal{O}_K/I)^*$, define the minimal norm on β by

$$N_I(\beta) := \min_{y\in\beta} |N_K(y)| \tag{8.1}$$

where N_K is the norm in the number field K. Define

$$L(K, I) := \max_{\beta\in(\mathcal{O}_K/I)^*} N_I(\beta). \tag{8.2}$$

The norm of an ideal I is $N(I) := |\mathcal{O}_K/I|$. Recall $\forall x \in I$, $N(I)|N_K(x)$.

In [15] it was proved that

Theorem 8.1. (Konyagin-Shparlinski) $L(K, \mathfrak{p}) = o(N(\mathfrak{p}))$ *for a sequence of prime ideals of asymptotic density 1 as* $N(\mathfrak{p}) \to \infty$.

Konyagin and Shparlinski also asked whether the same claim is true for almost all ideals. Recently this question was affirmatively answered by Bourgain and Chang [4]. They also improved the result regarding prime ideals by reducing the aymptotic size of the exceptional set.

Theorem 8.2. (Bourgain-Chang) $\forall\epsilon > 0$, $\exists\delta = \delta(\epsilon)$ *such that* $\lim_{\epsilon\to 0}\delta = 0$ *and*

$$L(K, I) \leq N(I)^{1-\epsilon} \tag{8.3}$$

holds for all ideals outside an exceptional collection Ω *of ideals of asymptotic density at most* δ *as* $N(I) \to \infty$.

The proofs of both Theorem 8.1 and Theorem 8.2 are based on exponential sums on finite fields.

Bourgain and Lindenstrauss observed that, assuming K is not a CM-field and $\mathrm{rank}(U_K) \geq 2$, a bound which is weaker than (8.3), but still $o(N(I))$, can be achieved by ergodic-theoretical methods for all ideals without exceptional set. Cerri made observations along the same lines in the study of Euclidean minima of number fields.

Such observations can be implemented using Proposition 7.2.

Theorem 8.3. *Suppose K is a non CM-number field of degree d with $\mathrm{rank}(U_K) \geq 2$. Then $L(K, I) = o(N(I))$ holds for all ideals I as $N(I) \to \infty$.*

More precisely, there are effective constants c_{21}, c_{22}, c_{23} depending only on d such that

$$\frac{L(K,I)}{N(I)} \leq \big(\log\log\log N(I)\big)^{-c_{21}\mathcal{F}^2_{U_K}} \tag{8.4}$$

for all ideals I with

$$\log\log\log N(I) \geq \big(c_{22}D(K)^{15(d-1)}, \max(\mathcal{F}_{U_K}, 2)^{c_{23}\mathcal{F}^2_{U_K}}\big). \tag{8.5}$$

Here $D(K)$ is the discriminant of K, and $\mathcal{F}_{U_K}$ is the logarithmic Mahler measure bound on some set that generates U_K up to finite index, as defined in Definition 2.3.

The group of units U_K acts on O_K by multiplication and preserves all ideals I. We know the embedding σ in (2.1) identifies $K \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{R}^d$ and embedds $\mathcal{O}_K$ as a full rank lattice. An ideal I is embedded as a sublattice of $\mathcal{O}_K \subset K \otimes_{\mathbb{Q}} \mathbb{R}$.

Recall the i-th coordinate y_i in $\mathbb{R}^d$ of $y \in K$ is $\sigma_i(y)$ for $1 \leq i \leq r_1$ and $y_{r_1+j} + \mathrm{i}y_{r_1+r_2+j} = \sigma_{r_1+j}(y)$ for $1 \leq j \leq r_1$. Thus

$$\begin{aligned} |N_K(y)| =& \prod_{i=1}^{d} |\sigma_i(y)| = \prod_{i=1}^{r_1} |y_i| \prod_{j=1}^{r_2} |y_{r_1+j} + \mathrm{i}y_{r_1+r_2+j}|^2 \\ \leq& (\max_{i=1}^{d} |y_i|)^{r_1} \big(2(\max_{i=1}^{d} |y_i|)^2\big)^{r_2} = 2^{r_2}(\max_{i=1}^{d} |y_i|)^d. \end{aligned} \tag{8.6}$$

Identify I with the lattice $\sigma(I)$ in $\mathbb{R}^d$. We are going to show that I doesn't degenerate. More precisely, for all I, there exists an isomorphism ψ between $\mathbb{T}^d$ and $(K \otimes_{\mathbb{Q}} \mathbb{R})/I$ whose uniformity $\mathcal{M}_\psi$ is bounded in terms of $D(K)$.

Lemma 8.4. *Let $C \subset \mathbb{R}^d$ be a closed convex set that is symmetric with respect to the origin and $\Lambda < R^d$ be a full rank lattice, $d \geq 2$. Denote by $m_1 \leq \cdots \leq m_d$ the successive minima of C with respect to Λ. Then there is a basis $w^1, \cdots w^n$ for Λ such that $w^i \in (\frac{3}{2})^{i-1} m_i C, \forall i$.*

Proof. By definition of successive minima, Λ contains d linearly independent vectors $v^1, \cdots, v^d$ such that $v^i \in m_i C$. They generate a finite-index sublattice of Λ.

For $i = 1, \cdots, d$, let L_i be the vector subspace spanned by $v^1, \cdots, v^i$ and $\Lambda_i = \Lambda \cap L_i$. Then Λ_i is a lattice of rank i and $\Lambda_d = \Lambda$.

We claim that there exists a set of vectors $w^1, \cdots, w^d$ such that for all i, $w^i \in \Lambda_i \cap (\frac{3}{2})^{i-1} m_i C$ and $w^1, \cdots, w^i$ form a basis of Λ_i. The proof is by induction.

When $i = 1$, $L_1 = \mathbb{R}v^1 \cong \mathbb{R}$ and Λ_1 is a discrete subgroup in it. Thus Λ_1 is cyclic. Suppose w^1 is a generator, then $v^1 = nw^1$ for some $n \in \mathbb{N}$ up to a change of sign. Hence $w^1 = \frac{1}{n}v^1 \in \frac{1}{n}m_d C \subset m_d C$, which proves the $i = 1$ case.

Suppose the claim is true for $i-1$. Consider the quotient group $\Lambda_i/\Lambda_{i-1} = \Lambda_i/(\Lambda_i \cap L_{i-1})$, which canonically embedds into $L_i/L_{i-1} \cong \mathbb{R}$ and hence is torsion-free. Moreover, its torsion free rank is $\operatorname{rank}(\Lambda_i) - \operatorname{rank}(\Lambda_{i-1}) = 1$. Thus $\Lambda_i/\Lambda_{i-1} \cong \mathbb{Z}$. Take $w \in \Lambda_i$ such that the coset $w + \Lambda_{i-1}$ is a generator. Notice since $v^i \notin L_{i-1}$, $v^i + \Lambda_{i-1}$ is a non-trivial coset. So $\forall v \in \Lambda_i$, up to a change of sign, $\exists n \in \mathbb{N}$ such that $n(w + \Lambda_{i-1}) = v^i + \Lambda_{i-1}$. In other words, $v = nw + \sum_{j=1}^{i-1} l_j w^j$ for some integers $l_1, \cdots, l_{i-1}$ or equivalently, $w^1, \cdots, w^{i-1}$ and w generate Λ_i.

In particular, if we take $v = v^i$, then $v^i = nw + \sum_{j=1}^{i-1} l_j w^j$ and $w = \frac{1}{n} v^i + \sum_{j=1}^{i-1} \frac{l_j}{n} w^j$ for some $n \in \mathbb{N}$ and $l_1, \cdots, l_{i-1} \in Z$. Decompose $\frac{l_j}{n} = a_j + k_j$ with $a_j \in [-\frac{1}{2}, \frac{1}{2}]$ and $k_j \in \mathbb{Z}$, then $w = w^i + \sum_{j=1}^{i-1} k_j w^j$ where $w^i = \frac{1}{n} v^i + \sum_{j=1}^{i-1} a_j w^j$. Hence $w^1, \cdots, w^{i-1}$ and w^i generate Λ_i. It suffices to show $w^i \in (\frac{3}{2})^i m_d C$.

As $v^i \in m_i C$ and $w_j \in (\frac{3}{2})^{j-1} m_j C, \forall j < i$,

$$
\begin{aligned}
w^i \in & \frac{1}{n} m_i C + \sum_{j=1}^{i-1} a_j (\frac{3}{2})^{j-1} m_j C \\
\subset & m_i C + \frac{1}{2} \sum_{j=1}^{i-1} (\frac{3}{2})^{j-1} m_i C = \big(1 + \frac{1}{2} \sum_{j=1}^{i-1} (\frac{3}{2})^{j-1}\big) m_i C \\
= & (\frac{3}{2})^{i-1} m_i C.
\end{aligned}
$$

Thus the claims is proved. Let $i = d$, the lemma follows. □

Lemma 8.5. *There exists an isomorphim $\psi : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $\psi(\mathbb{Z}^d) = I$, $\mathcal{M}_\psi \lesssim_d D(K)^{\frac{d-1}{2d}}$.*

Proof. Define a closed symmetric convex set $C = \{z \in \mathbb{R}^d \big| \max_{i=1}^d |z_i| \le 1\}$ in $\mathbb{R}^d$, whose volume is 2^d. Suppose $m_1 \le \cdots \le m_d$ are the succesive minima of C with respect to the lattice I. Then by Minkowski's Second Theorem:

$$m_1^{d-1} m_d \le m_1 m_2 \cdots m_d \lesssim_d \operatorname{covol}(I) = D(K)^{\frac{1}{2}} N(I). \tag{8.7}$$

On the other hand, by the definition of successive minima, $\exists y \in I \cap m_1 C$, $y \neq 0$. Then $|y_i| \le m_1$, $\forall i$. By (8.6)

$$|N_K(y)| \lesssim_d m_1^d. \tag{8.8}$$

Notice $N(I) | N_K(y)$ and $N_K(y) \neq 0$, thus $N(I) \le |N_K(y)|$. Compare (8.7) and (8.8), we see

$$m_d \lesssim_d {m_1}^{-(d-1)} D(K)^{\frac{1}{2}} N(I) \lesssim_d |N_K(y)|^{-\frac{d-1}{d}} D(K)^{\frac{1}{2}} N(I) \le D(K)^{\frac{1}{2}} N(I)^{\frac{1}{d}}. \tag{8.9}$$

and for all k,

$$
\begin{aligned}
\prod_{\substack{i=1,\cdots,d \\ i \neq k}} m_i & \lesssim_d {m_1}^{-1} D(K)^{\frac{1}{2}} N(I) \lesssim_d |N_K(y)|^{-\frac{1}{d}} D(K)^{\frac{1}{2}} N(I) \\
& \le D(K)^{\frac{1}{2}} N(I)^{\frac{d-1}{d}}.
\end{aligned} \tag{8.10}
$$

The previous lemma says that there is basis $y^1, \cdots, y^d$ of I such that

$$\max_{j=1}^d |y_j^i| \lesssim_d m_i, \forall i = 1, \cdots, d, \tag{8.11}$$

where y^i_j is the j-th coordinate of y^i.

Define a $d \times d$ matrix $\psi := (y^i_j)$, then $\psi(\mathbb{Z}^n) = I$. In particular

$$|\det \psi| = \mathcal{S}^d_I = \mathrm{covol}(I) = D(K)^{\frac{1}{2}} N(I). \tag{8.12}$$

From (8.9) and (8.11) it is easy to see

$$\|\psi\| \lesssim_d m_d \lesssim_d D(K)^{\frac{1}{2}} N(I)^{\frac{1}{d}}. \tag{8.13}$$

Denote ψ^{-1} by (a^j_i), then each entry $a^i_j = \frac{A^j_i}{|\det \psi|}$ where A^j_i is the determinant of some $(d-1)\times(d-1)$-submatrix in ψ. By (8.10) and (8.11) we obtain $|A^i_j| \lesssim_d D(K)^{\frac{1}{2}} N(I)^{\frac{d-1}{d}}$, so

$$\|\psi^{-1}\| \lesssim_d \max_{i,j} |a^i_j| \lesssim_d \frac{D(K)^{\frac{1}{2}} N(I)^{\frac{d-1}{d}}}{D(K)^{\frac{1}{2}} N(I)} = N(I)^{-\frac{1}{d}}. \tag{8.14}$$

Combining (8.12), (8.13) and (8.14), we obtain

$$\begin{aligned}
\mathcal{M}_\psi =& \max(\|\psi\| \mathcal{S}^{-1}_I, \|\psi^{-1}\| \mathcal{S}_I) \\
&\lesssim_d \max\left(D(K)^{\frac{1}{2}} N(I)^{\frac{1}{d}} \big(D(K)^{\frac{1}{2}} N(I)\big)^{-\frac{1}{d}}, N(I)^{-\frac{1}{d}} \big(D(K)^{\frac{1}{2}} N(I)\big)^{\frac{1}{d}} \right) \\
&= D(K)^{\frac{d-1}{2d}}.
\end{aligned}$$

□

Now we are ready to prove the theorem.

Proof of Theorem 8.3. As $U_K = \mathcal{O}^*_K$ preserves I. U_K acts as an automorphism group of the twisted torus $X = (\mathbb{K} \otimes_{\mathbb{Q}} \mathbb{R})/I$, moreover U_K is volume preserving as $\forall u \in U_K$, $|\det(\times_u)| = |N(u)| = 1$. Thus using the isomorphism ψ constructed in Lemma 8.5 as a conjugation map, the U_K action is conjugate to the G-action on $\mathbb{T}^d$ where $G \cong U_K$ is an abelian subgroup in $SL_d(\mathbb{Z})$. In other words, there is a group isomorphism $\phi : G \overset{\sim}{\mapsto} U_K$ such that $g.x = \psi^{-1}\big(\times_u .\psi(x)\big), \forall x \in \mathbb{T}^d$.

So G satisfies Condition 2.8 with $\Gamma = I$ as we assumed K is not CM and $\mathrm{rank}(U_K) \geq 2$.

$\beta \in (\mathcal{O}_K/I)^* \subset \mathcal{O}_K/I$ is a rational point in the twisted torus X. Let us study its orbit $U_K.\beta \subset X$ under the U_K action. Denote the stabilizer in U_K of β by U^β_K. Suppose $u \in U^\beta_K$, then $u.\beta = \beta$. Because β is invertible in the ring $\mathcal{O}_K/I$, this implies $u.(1+I) = (1+I)$ or equivalently $u \in 1+I$. This helps us to estimate the size of u. Notice

$$\begin{aligned}
2^{h^{\mathrm{Mah}}(u)} =& \prod_{i=1}^d \max(|\sigma_i(u)|, 1) \geq \prod_{i=1}^d \frac{|\sigma_i(u)|+1}{2} \geq 2^{-d} \prod_{i=1}^d (|\sigma_i(u) - 1|) \\
\geq& 2^{-d} \prod_{i=1}^d (|\sigma_i(u-1)|) = 2^{-d} |N(u-1)|.
\end{aligned}$$

However as $u - 1 \in I$, if $u \neq 1$ then $|N(u-1)| \geq N(I)$ thus its logarithmic Mahler measure is bounded from below: $h^{\mathrm{Mah}}(u) \geq \log(2^{-d} N(I))$.

Let $\mathcal{F}_{U_K}$ be defined by Definition 2.3. Then there are d multiplicatively independent elements $u_1, \cdots, u_r \in U_K$ such that $h^{\mathrm{Mah}}(u_i) \leq \mathcal{F}_{U_K}$ where $r = \mathrm{rank}(U_K)$. Consider $Z = \{\prod_{i=1}^r u_i^{n_i} | 0 \leq n_i <$

$\frac{1}{d}\mathcal{F}_{U_K}^{-1}\log(2^{-d}N(I))\}$, then it is not hard to see that for any pair $u \neq u'$ inside Z, $h^{\mathrm{Mah}}(u^{-1}u') < \log(2^{-d}N(I))$ so $u^{-1}u' \notin U_K^\beta$. Therefore $|U_K.\beta| = |U_K/U_K^\beta| \geq |Z| \gtrsim_d \mathcal{F}_{U_K}^{-r}\big(\log(2^{-d}N(I))\big)^r$, where the last inequality relies on assumption (8.5).

Let $x = \psi^{-1}(\beta) \in \mathbb{T}^d$, then its orbit is $G.x = \psi^{-1}(U_K.\beta)$, thus

$$|G.x| \gtrsim_d \mathcal{F}_{U_K}^{-r}\big(\log(2^{-d}N(I))\big)^r. \tag{8.15}$$

At this point we may deduce the follow claim:

Claim *There are effective constants $\iota_1(d)$, $\iota_2(d)$ and $c_{23}(d)$ such that if*

$$\log\log\log N(I) \geq \big(\iota_2\mathcal{M}_\psi^{30d}, \max(\mathcal{F}_{U_K}, 2)^{c_{23}\mathcal{F}_{U_K}^2}\big) \tag{8.16}$$

then the orbit $G.x$ is $\big(\log\log\log N(I)\big)^{-\iota_1\mathcal{F}_{U_K}^2}$-dense.

The claim is proved by applying Proposition 7.2 to the set $E = G.x$. The proof is almost identical to that of Proposition 7.6, based on (8.15) instead of on (7.27) (notice $\mathcal{F}_{\phi(G)} = \mathcal{F}_{U_K}$); so we are not going to elaborate here.

By Lemma 8.5, when c_{22} is large enough then assumption (8.16) follows from the condition (8.5) in the theorem so the claims applies and $U_K.\beta = \psi(G.x)$ is ρ_0-dense in X where $\rho_0 = \|\psi\|\big(\log\log\log N(I)\big)^{-\iota_1\mathcal{F}_{U_K}^2}$. This implies that $\exists y \in \beta$ and $u \in U_K$ such that $\sigma(uy) \in \mathbb{R}^d$ is within distance ρ_0 from the origin. By (8.6) and (8.13) we have

$$\begin{aligned} N_I(\beta) \leq& |N_K(y)| = |N_K(uy)| \\ &\lesssim_d \rho_0^d = \|\psi\|^d\big(\log\log\log N(I)\big)^{-d\iota_1\mathcal{F}_{U_K}^2} \\ &\leq D(K)^{\frac{d}{2}}N(I)\big(\log\log\log N(I)\big)^{-d\iota_1\mathcal{F}_{U_K}^2}. \end{aligned} \tag{8.17}$$

As $\log\log\log N(I) \gtrsim_d D(K)^{15(d-1)}$ by assumption and $F_{U_K} \gtrsim_d 1$, whenever c_{21}, c_{22}, c_{23} in the statement of the theorem are made sufficiently large, in the inequality above both $D(K)^{\frac{d}{2}}$ and the implied constant can be absorbed into the last factor, which gives

$$N_I(\beta) \leq N(I)\big(\log\log\log N(I)\big)^{-c_{21}\mathcal{F}_{U_K}^2}.$$

(8.4) follows. $\square$

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